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https://europroofnet.github.io/MCLP/

# On Natural Deduction and Axioms for Propositional and First-Order Logic

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**DTU Compute is Denmark's largest mathematics and computer science environment**

***Thanks to my student Vladimir Kalabukhov for help with proofs***

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- **Proof of a so-called grandfather formula using natural deduction**
- **Proof of $p \longrightarrow p$ from standard axioms on paper and in Isabelle**
- **From propositional logic to implicational logic**
- **Proof of soundness and completeness for implicational logic**
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- **BCD-logic**
- **Formalizations L1 & L2**
- **Formalizations L1A & L2A (Alternatives)**
- **Conclusions**

**But first: Prelude**

# Grandfather

**Natural Language Reasoning Example**

A really difficult example is discussed on page 128 of the Handbook of Tableau Methods (Kluwer Academic Publishers 1999):

> *If every person that is not rich has a rich father, then some rich person must have a rich grandfather.*

Formalization with $r$ (*rich*) and $f$ (*father*): $\forall x(\neg r(x) \to r(f(x))) \to \exists x(r(x) \wedge r(f(f(x))))$

```
proposition ‹(∀x. ¬ r x ⟶ r (f x)) ⟶ (∃x. r x ∧ r (f (f x)))›
  by auto
```

**... or by blast, by meson or by metis in Isabelle/HOL**

# Why Isabelle for 'MathCompLing' Proofs 1985-2025

Formalizing 1000+ Theorems https://1000-plus.github.io/

Currently 1198 Theorems (Wikipedia)

https://en.wikipedia.org/wiki/Gödel's_completeness_theorem
https://en.wikipedia.org/wiki/Gödel's_incompleteness_theorem

Currently 6 Proof Assistants (16 September 2025)

Formalizing 100 Theorems https://www.cs.ru.nl/~freek/100/

| Higher-Order Logic | | Type Theory | | Set Theory | |
|---|---|---|---|---|---|
| Isabelle | 92 | Lean | 82 | Metamath | 74 |
| HOL Light | 89 | Rocq (Coq) | 79 | Mizar | 71 |

All 100 Formalized Except Fermat's Last Theorem

## Isabelle/HOL – Classical Higher-Order Logic

## Natural Deduction

| | |
|---|---|
| `Imp_I:  (p ⟹ q) ⟹ p ⟶ q` | `Imp_E:  p ⟶ q ⟹ p ⟹ q` |
| `Uni_I:  (⋀x. p x) ⟹ ∀x. p x` | `Uni_E:  ∀x. p x ⟹ p c` |

```
Iff_I:        (p ⟹ q) ⟹ (q ⟹ p) ⟹ p = q

Extension:    (⋀x. f x = g x) ⟹ f = g

Choice:       p c ⟹ p (ε p)
```

**The usual logical operators can be defined and introduction/elimination rules proved**

# A Novel Axiomatics for Implicational Logic



One axiom and five rules

AT $(q \to r) \to (r \to p) \to q \to p$

CR $q \to r \to p \implies r \to q \to p$

DR $q \to p \implies r \to q \to p$

MP $q \to p \implies q \implies p$

IR $(q \to q) \to p \implies p$

PR $(p \to q) \to p \implies p$

AT & MP from Łukasiewicz

CR & IR both added give BCI logic

PR also added gives classical logic

BCI/BCK logics by Meredith

DR also added gives BCK logic

PR = Peirce's Rule

# Recent Papers

*Minimal Sequent Calculus for Teaching First-Order Logic: Lessons Learned*

Jørgen Villadsen

ThEdu 75-89 2024

*A Sequent Calculus for First-Order Logic Formalized in Isabelle/HOL*

Asta Halkjær From, Anders Schlichtkrull & Jørgen Villadsen

Journal of Logic and Computation 33 818-836 2023

*Teaching Higher-Order Logic Using Isabelle*

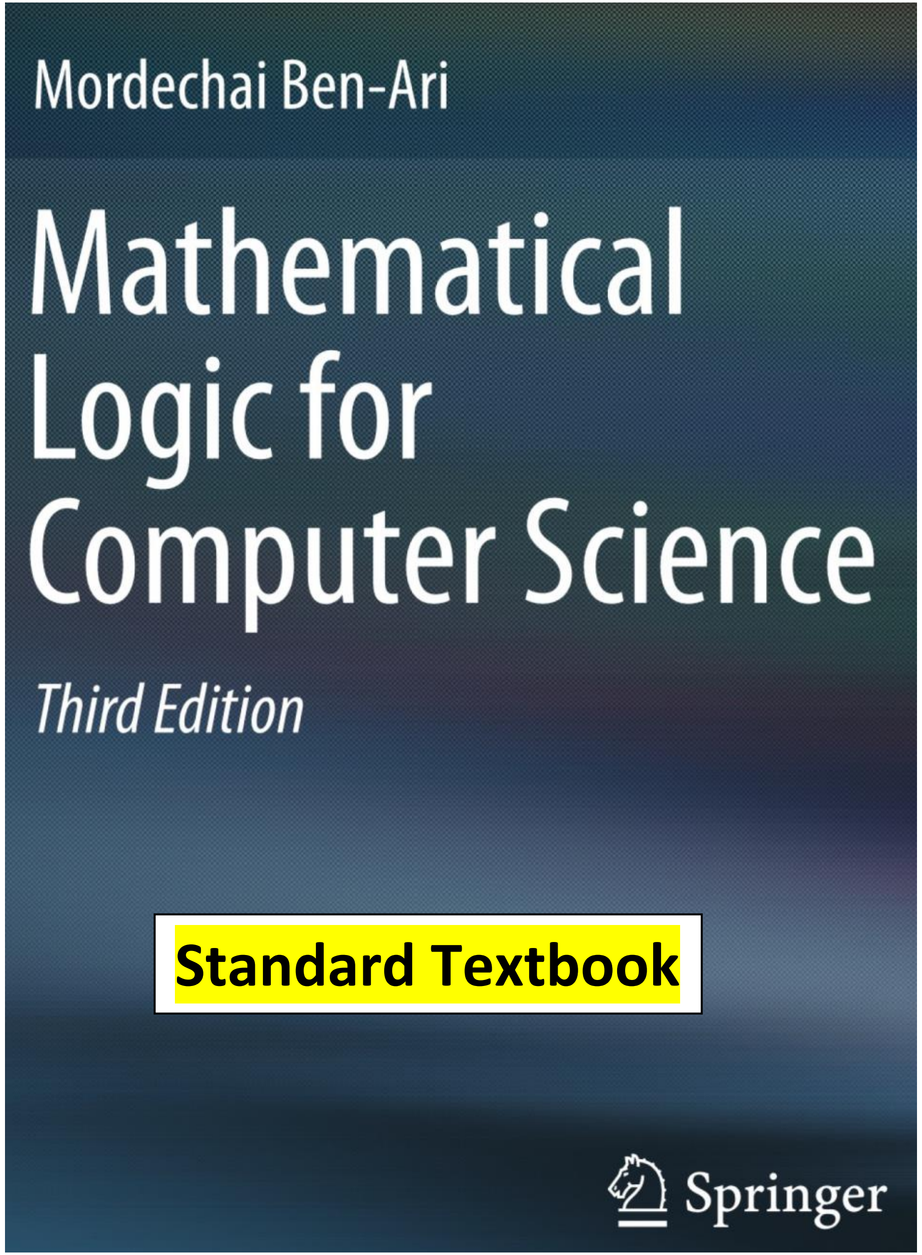
Simon Tobias Lund & Jørgen Villadsen

ThEdu 59-78 2023

*ProofBuddy: A Proof Assistant for Learning and Monitoring*

Nadine Karsten, Frederik Krogsdal Jacobsen, Kim Jana Eiken, Uwe Nestmann & Jørgen Villadsen

TFPIE 1-21 2023




**Standard Textbook**

**Imp (Uni (Con p q[0])) (Con q[a] p)**

```
Imp_R
  Neg (Uni (Con p q[0]))
  Con q[a] p
Uni_L
  Neg (Con p q[a])
  Con q[a] p
Con_L
  Neg p
  Neg q[a]
  Con q[a] p
Ext
  Con q[a] p
  Neg p
  Neg q[a]
Con_R
  q[a]
  Neg p
  Neg q[a]
+
  p
  Neg p
  Neg q[a]
Basic
```

**Minimal Sequent Calculus**

## MiniCalc Web App

**Isabelle Checks Proofs**

**Example:**

$(\forall x.\ p \land q\ x) \longrightarrow q\ a \land p$

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```
notation (input) False (‹⊥›) and True (‹T›)

theorem Imp_C: ‹(p ⟶ q ⟹ p) ⟹ p›
  using ccontr by iprover

lemma LEM: ‹p ∨ ¬ p›
proof (rule Imp_C)
  assume ‹p ∨ ¬ p ⟶ ⊥›
  have ‹¬ p›
  proof
    assume p
    then have ‹p ∨ ¬ p› ..
    with ‹p ∨ ¬ p ⟶ ⊥› show ⊥ ..
  qed
  then show ‹p ∨ ¬ p› ..
qed
```

```
proposition ‹(∀x. ¬ r x ⟶ r (f x)) ⟶ (∃x. r x ∧ r (f (f x)))›

proof
  assume SECRET
  fix SECRET
  from SECRET have child SECRET
  have SECRET by (rule LEM)
  then have
  proof
    assume SECRET
    then show SECRET
  next
    assume SECRET
    with child have SECRET
    then show SECRET
  qed
  then obtain SECRET where SECRET
  from SECRET have father SECRET
  have SECRET by (rule LEM)
  then show SECRET
  proof
    assume SECRET
    from SECRET have grandfather SECRET
    have SECRET by (rule LEM)
    then show SECRET
    proof
      assume SECRET
      from SECRET and SECRET have SECRET
      then show SECRET
    next
      assume SECRET
      with grandfather have SECRET
      from SECRET and SECRET have SECRET
      then show SECRET
    qed
  next
    assume SECRET
    from father and SECRET have SECRET
    with SECRET have SECRET
    then show SECRET
  qed
qed
```

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| | |
|---|---|
| **Axiom 1** | $\vdash (A \to (B \to A))$ |
| **Axiom 2** | $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$ |
| **Axiom 3** | $\vdash (\neg B \to \neg A) \to (A \to B)$ |

The *rule of inference* is ***modus ponens*** (*MP* for short):

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B}$$

*Proof*

| | | |
|---|---|---|
| 1. | $\vdash (A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$ | Axiom 2 |
| 2. | $\vdash A \to ((A \to A) \to A)$ | Axiom 1 |
| 3. | $\vdash (A \to (A \to A)) \to (A \to A)$ | MP 1, 2 |
| 4. | $\vdash A \to (A \to A)$ | Axiom 1 |
| 5. | $\vdash A \to A$ | MP 3, 4 |

Axiom 1: $p \longrightarrow q \longrightarrow p$ Axiom 2: $(p \longrightarrow q \longrightarrow r) \longrightarrow (p \longrightarrow q) \longrightarrow p \longrightarrow r$

| | | |
|---|---|---|
| 1. | $(p \longrightarrow (p \longrightarrow p) \longrightarrow p) \longrightarrow (p \longrightarrow p \longrightarrow p) \longrightarrow p \longrightarrow p$ | Axiom 2 |
| 2. | $p \longrightarrow (p \longrightarrow p) \longrightarrow p$ | Axiom 1 |
| 3. | $(p \longrightarrow p \longrightarrow p) \longrightarrow p \longrightarrow p$ | MP |
| 4. | $p \longrightarrow p \longrightarrow p$ | Axiom 1 |
| 5. | $p \longrightarrow p$ | MP |

```
lemma S: ‹(p ⟶ q ⟶ r) ⟶ (p ⟶ q) ⟶ p ⟶ r› and K: ‹p ⟶ q ⟶ p›
  by auto

lemma I: ‹p ⟶ p›
  using mp mp S K K .
```

The same proof in Isabelle

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# Łukasiewicz's Axioms for Propositional Logic

$$(q \longrightarrow r) \longrightarrow (r \longrightarrow p) \longrightarrow q \longrightarrow p$$
$$(\neg\, p \longrightarrow p) \longrightarrow p$$
$$q \longrightarrow \neg\, q \longrightarrow p$$

*Elements of Mathematical Logic*
Jan Łukasiewicz, Professor at Warsaw University
Authorized lecture notes prepared by M. Presburger
1929

We do not use negation as a primitive operator, but define negation in terms of implication and falsity like in Isabelle/HOL: $\neg\, p \;\equiv\; p \longrightarrow \bot$

But $(q \longrightarrow r) \longrightarrow (r \longrightarrow p) \longrightarrow q \longrightarrow p$ is not a single axiom for implicational logic

# Bernays-Tarski's Axioms for Implicational Logic

$p \longrightarrow q \longrightarrow p$
$((p \longrightarrow q) \longrightarrow p) \longrightarrow p$
$(q \longrightarrow r) \longrightarrow (r \longrightarrow p) \longrightarrow q \longrightarrow p$

Second axiom is Peirce's law

https://en.wikipedia.org/wiki/Peirce's_law

See also

https://en.wikipedia.org/wiki/Implicational_propositional_calculus

Perhaps Tarski-Bernays is a more appropriate name for the axiomatics

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# Implicational Logic

```
datatype 'a form =
  Pro 'a (‹¤›) |
  Imp ‹'a form› ‹'a form› (infixr ‹→› 55)

primrec semantics (infix ‹⊨› 50) where
  ‹I ⊨ ¤n = I n› |
  ‹I ⊨ p → q = (I ⊨ p ⟶ I ⊨ q)›

inductive Ax (‹⊢ _› 50) where
  AK: ‹⊢ p → q → p› |
  AP: ‹⊢ ((p → q) → p) → p› |
  AT: ‹⊢ (p → q) → (q → r) → p → r› |
  MP: ‹⊢ p → q ⟹ ⊢ p ⟹ ⊢ q›

theorem soundness: ‹⊢ p ⟹ I ⊨ p›
  by (induct p rule: Ax.induct) auto

lemma AC: ‹⊢ (p → q → r) → q → p → r›
  using MP MP AT AT MP AT MP MP AT MP MP AT AK AT MP MP AT AT AP .

lemma AS: ‹⊢ (p → q → r) → (p → q) → p → r›
  using MP MP AT MP MP AT AC MP AC AT MP MP AC AT MP MP AT AT AP .

lemma AI: ‹⊢ p → p›
  using MP MP AS AK AK .
```

```
theorem soundness:     ‹⊢ p ⟹ I ⊨ p›

theorem completeness: ‹∀I. I ⊨ p ⟹ ⊢ p›

theorem main:          ‹(⊢ p) = (∀I. I ⊨ p)›
  using
    soundness and completeness by auto
```

**Isabelle - 175 lines 1 second**

**Alternatively in one go with 73 axioms and rules:**

```
lemma AS: ‹⊢ (q → r → p) → (q → r) → q → p›
  using MP MP AT MP MP AT MP MP AT AT MP AT MP MP AT MP MP AT AK AT
        MP MP AT AT AP MP MP MP AT AT MP AT MP MP AT MP MP AT AK AT
        MP MP AT AT AP AT MP MP MP MP AT AT MP AT MP MP AT MP MP AT
        AK AT MP MP AT AT AP AT MP MP AT AT AP .
```

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**Standard Textbook**

**Axiom 1** $\vdash (A \to (B \to A))$

**Axiom 2** $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

**Axiom 3** $\vdash (\neg B \to \neg A) \to (A \to B)$

**Axiom 4** $\vdash \forall x A(x) \to A(a)$

**Axiom 5** $\vdash \forall x (A \to B(x)) \to (A \to \forall x B(x))$

- In Axioms 1, 2 and 3, $A$, $B$ and $C$ are any formulas of first-order logic.
- In Axiom 4, $A(x)$ is a formula with a free variable $x$.
- In Axiom 5, $B(x)$ is a formula with a free variable $x$, while $x$ is *not* a free variable of the formula $A$.

The rules of inference are ***modus ponens*** and ***generalization***:

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \qquad \frac{\vdash A(a)}{\vdash \forall x A(x)}$$

# Axiomatics for First-Order Logic

**We present formalizations in the proof assistant Isabelle/HOL of axiomatics for classical first-order logic, based on natural deduction, where the soundness and completeness theorems hold for languages of arbitrary cardinalities.**

$$(r \longrightarrow p) \longrightarrow (q \longrightarrow r) \longrightarrow q \longrightarrow p$$
$$(q \longrightarrow r \longrightarrow p) \longrightarrow r \longrightarrow q \longrightarrow p$$
$$(q \longrightarrow p) \longrightarrow r \longrightarrow q \longrightarrow p$$
$$q \longrightarrow p \Longrightarrow q \Longrightarrow p$$
$$(q \longrightarrow q) \longrightarrow p \Longrightarrow p$$
$$(p \longrightarrow q) \longrightarrow p \Longrightarrow p$$
$$q \longrightarrow \mathrm{False} \Longrightarrow q \longrightarrow p$$
$$q \longrightarrow (\forall x.\ p'\ x) \Longrightarrow q \longrightarrow p'\ t$$
$$(\bigwedge x.\ q \longrightarrow p'\ x) \Longrightarrow q \longrightarrow (\forall x.\ p'\ x)$$

**Key result**
**BCD-Logic**

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# BCD-Logic

I $p \to p$

K $p \to q \to p$

B $(q \to r) \to (r \to p) \to q \to p$

C $(q \to r \to p) \to r \to q \to p$

D $(q \to p) \to r \to q \to p$

**BCI**

**BCK = IBCD**

**All weaker than intuitionistic and classical logic**

$$p \longrightarrow q \longrightarrow p$$
$$((p \longrightarrow q) \longrightarrow p) \longrightarrow p$$
$$(q \longrightarrow r) \longrightarrow (r \longrightarrow p) \longrightarrow q \longrightarrow p$$
$$\text{False} \longrightarrow p$$
$$(\forall x.\ p'\ x) \longrightarrow p'\ t$$

$$q \longrightarrow p \Longrightarrow q \Longrightarrow p$$

$$(\bigwedge x.\ q \longrightarrow p'\ x) \Longrightarrow q \longrightarrow (\forall x.\ p'\ x)$$

<u>**Axiom Replacements**</u>

**BCD Axioms**

$$(r \longrightarrow p) \longrightarrow (q \longrightarrow r) \longrightarrow q \longrightarrow p$$
$$(q \longrightarrow r \longrightarrow p) \longrightarrow r \longrightarrow q \longrightarrow p$$
$$(q \longrightarrow p) \longrightarrow r \longrightarrow q \longrightarrow p$$

**Identity Rule**

**Peirce's Rule**

$$(q \longrightarrow q) \longrightarrow p \Longrightarrow p$$
$$(p \longrightarrow q) \longrightarrow p \Longrightarrow p$$

**Explosion Rule**

**Specialization Rule**

$$q \longrightarrow \text{False} \Longrightarrow q \longrightarrow p$$
$$q \longrightarrow (\forall x.\ p'\ x) \Longrightarrow q \longrightarrow p'\ t$$

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# L1 – 3 Axioms / 6 Rules – Formal Soundness and Completeness Theorems

```
AB: ‹⊢ (r ⟶ p) ⟶ (q ⟶ r) ⟶ q ⟶ p›
AC: ‹⊢ (q ⟶ r ⟶ p) ⟶ r ⟶ q ⟶ p›
AD: ‹⊢ (q ⟶ p) ⟶ r ⟶ q ⟶ p›
MP: ‹⊢ q ⟶ p ⟹ ⊢ q ⟹ ⊢ p›
IR: ‹⊢ (q ⟶ q) ⟶ p ⟹ ⊢ p›
PR: ‹⊢ (p ⟶ q) ⟶ p ⟹ ⊢ p›
XR: ‹⊢ q ⟶ ⊥ ⟹ ⊢ q ⟶ p›
SR: ‹⊢ q ⟶ ∀p ⟹ ⊢ q ⟶ ⟨t⟩p›
GR: ‹⊢ q ⟶ ⟨⋆a⟩p ⟹ ⊢ q ⟶ ∀p› if ‹new a p› and ‹new a q›
```

**BCD-logic**

# L2 – 5 Axioms / 2 Rules – Formal Soundness and Completeness Theorems

AK: ‹⊩ p ⟶ q ⟶ p›
AP: ‹⊩ ((p ⟶ q) ⟶ p) ⟶ p›
AT: ‹⊩ (q ⟶ r) ⟶ (r ⟶ p) ⟶ q ⟶ p›
AX: ‹⊩ ⊥ ⟶ p›
AY: ‹⊩ ∀p ⟶ ⟨t⟩p›
MP: ‹⊩ q ⟶ p ⟹ ⊩ q ⟹ ⊩ p›
GR: ‹⊩ q ⟶ ⟨⋆a⟩p ⟹ ⊩ q ⟶ ∀p› **if** ‹new a p› **and** ‹new a q›

**Bernays-Tarski's axioms for classical implicational logic**

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# L1A – 4 Axioms / 2 Rules – Formal Soundness and Completeness Theorems

```
AK: ‹⊢ p ⟶ q ⟶ p›
AS: ‹⊢ (q ⟶ r ⟶ p) ⟶ (q ⟶ r) ⟶ q ⟶ p›
AY: ‹⊢ ∀p ⟶ ⟨t⟩p›
DN: ‹⊢ ¬ ¬ p ⟶ p›
MP: ‹⊢ q ⟶ p ⟹ ⊢ q ⟹ ⊢ p›
GR: ‹⊢ q ⟶ ⟨⋆a⟩p ⟹ ⊢ q ⟶ ∀p› if ‹new a p› and ‹new a q›
```

**Simple double negation axiom added to intuitionistic logic**

# L2A – 1 Axiom / 8 Rules – Formal Soundness and Completeness Theorems

```
AT: ‹⊩ (q ⟶ r) ⟶ (r ⟶ p) ⟶ q ⟶ p›
MP: ‹⊩ q ⟶ p ⟹ ⊩ q ⟹ ⊩ p›
CR: ‹⊩ q ⟶ r ⟶ p ⟹ ⊩ r ⟶ q ⟶ p›
DR: ‹⊩ q ⟶ p ⟹ ⊩ r ⟶ q ⟶ p›
IR: ‹⊩ (q ⟶ q) ⟶ p ⟹ ⊩ p›
PR: ‹⊩ (p ⟶ q) ⟶ p ⟹ ⊩ p›
XR: ‹⊩ q ⟶ ⊥ ⟹ ⊩ q ⟶ p›
SR: ‹⊩ q ⟶ ∀p ⟹ ⊩ q ⟶ ⟨t⟩p›
GR: ‹⊩ q ⟶ ⟨⋆a⟩p ⟹ ⊩ q ⟶ ∀p› if ‹new a p› and ‹new a q›
```

**BCD-logic variant closer to natural deduction**

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## Conclusions

For more than five years we have used the Isabelle proof assistant to teach metatheory of propositional and first-order logic, not only for natural deduction and sequent calculus, but also for axiomatic systems.

Languages of any cardinality are supported.

The focus is on the formal soundness and completeness theorems for classical first-order logic.

The formalizations consist of about 2000 lines of proof in Isabelle/HOL for L1/L2 as well as for L1A/L2A and the verification takes about 10 seconds.