Extending Flexible Boolean Semantics for the Language of Mathematics

Investigating the Semantics of Mathematical Language

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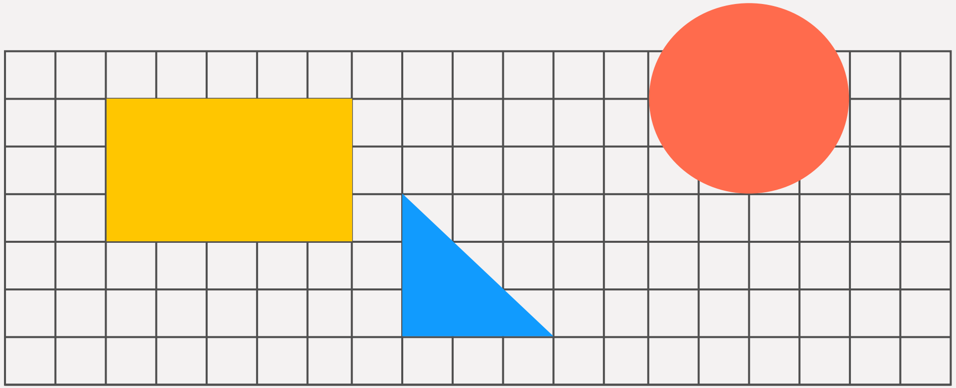

MANCHESTER 1824
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A Mathematico-Linguistic Puzzle

(1) A. x and y are prime.
    B. x is prime.

(2) A. x and y are coprime.
    B. # x is coprime.

(3) A. P and Q are countable.
    B. P is countable.

(4) A. P and Q are equinumerous.
    B. # P is equinumerous.

Why does A entail B in (1) and (3), but not in (2) and (4)?

1. Linguistic Phenomena in Question
2. Flexible Boolean Semantics and Problems
3. Winter’s Solution and its Problems
4. A Typing Fix

Linguistic Phenomena in Question

X and Y are P

Definite Descriptions

The integers
The set X

Coordination

2 and 3,
The integers and the rationals,
The set X and the set Y

Distributivity
Prime, countable

Collectivity
Coprime, equinumerous

Selectional Restrictions

Countable, equinumerous

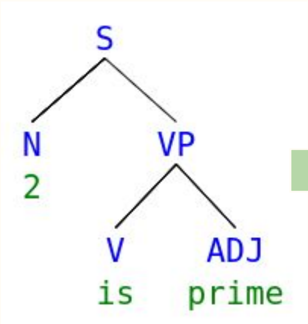
Prime, coprime

Flexible Boolean Semantics and Problems

Montague Semantics: Building Meaning from Parts

2 is prime

Parsing



Semantic Composition

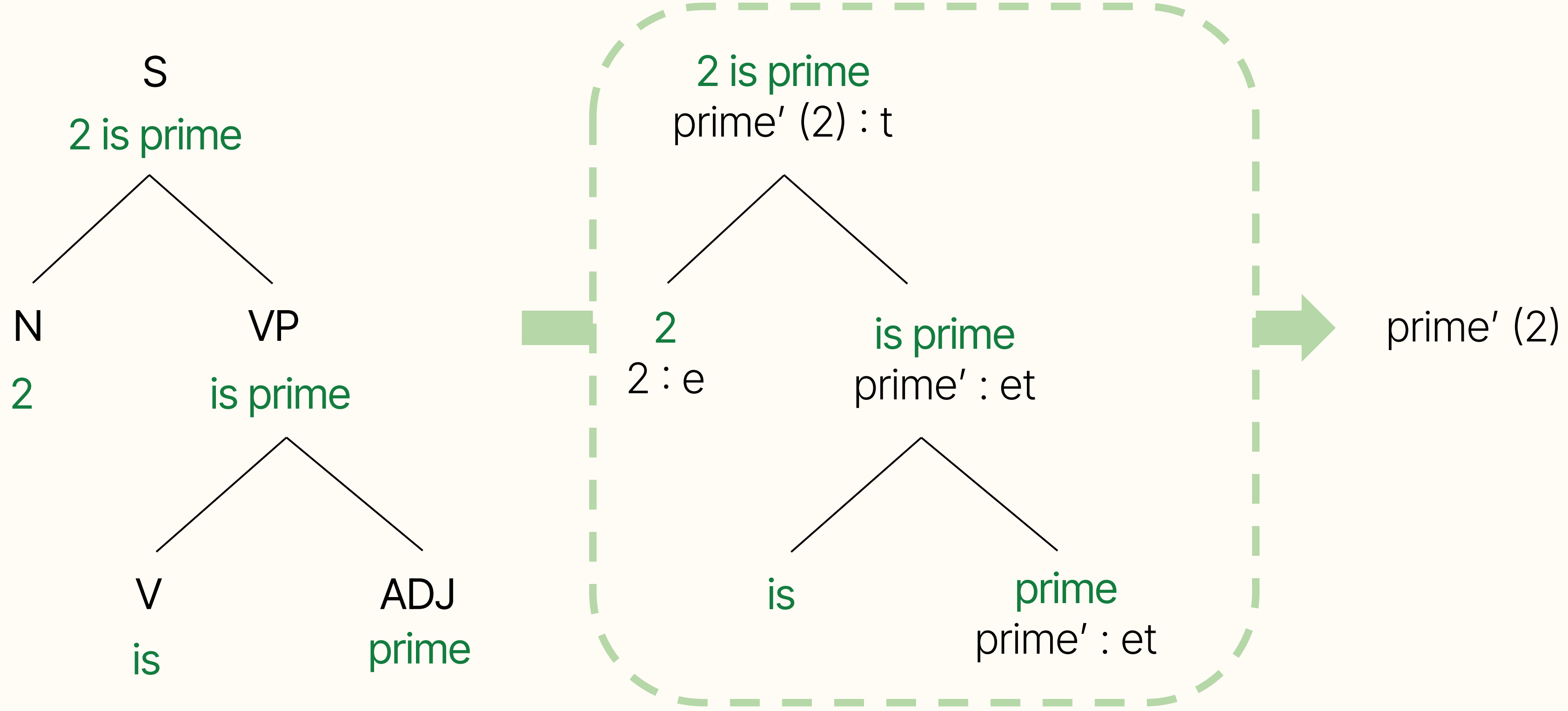
prime′(2)

Montague Semantics in Action: Types

e t et tt e(et) (et)t ...

Linguistic Expression	Type
2 is prime	t
2	e
prime	et

Montague Semantics in Action: Composition



Boolean Extensions of Montague Semantics

Sentence	Syntactic Category Coordinated	Types
2 is prime and **3 is prime**	Sentence	t
2 and **3** are prime	Noun Phrase	e
2 is **prime** and **even**	Adjective	et

Boolean Semantics:

Domains = Boolean algebras

and = meet

$\wedge$ for t

$\cap$ for σt

$[\![2 \text{ is prime and } 3 \text{ is prime}]\!]$
$= [\![2 \text{ is prime}]\!]_t \wedge [\![3 \text{ is prime}]\!]_t$
$= \text{prime}'(2) \wedge \text{prime}'(3)$

$[\![2 \text{ is prime and even}]\!]$
$= [\![\text{prime and even}]\!]_{et}\, [\![2]\!]_e$
$= 2 \in [\![\text{prime and even}]\!]_{et}$
$= 2 \in \text{prime}' \cap \text{even}'$

Boolean Operation on Entities?

Sentence	Syntactic Category Coordinated	Type
2 and **3** are prime	Noun Phrase	e

Boolean Semantics:

Domains = Boolean algebras

And = meet

$\wedge$ for t

$\cap$ for σt

??? for e

$$\boxed{\mathbf{M}_{e(et)t} = \lambda x_e \{ P_{et} \mid P(x) \}}$$

$$[\![2 \text{ and } 3]\!]_{(et)t} = \mathbf{M}(2)_{(et)t} \cap \mathbf{M}(3)_{(et)t}$$

$$= \{ P \mid P(2) \} \cap \{ Q \mid Q(3) \}$$

Winter, Y.: Flexibility Principles in Boolean Semantics: The Interpretation of Coordination, Plurality, and Scope in Natural Language, chap. Coordination and Collectivity. The MIT Press (02 2002).

Flexible Boolean Semantics

$$\boxed{\mathbf{M}_{e(et)t} = \lambda x_e \{ P_{et} \mid P(x) \}}$$

$$\begin{aligned} [\![2 \text{ and } 3]\!]_{(et)t} &= \mathbf{M}(2) \cap \mathbf{M}(3) \\ &= \{ P \mid P(2) \} \cap \{ Q \mid Q(3) \} \end{aligned}$$

$$\begin{aligned} [\![2 \text{ and } 3 \text{ are prime}]\!] &= \text{prime}' \in [\![2 \text{ and } 3]\!]_{(et)t} \\ &= \text{prime}' \in \{ P \mid P(2) \} \cap \{ Q \mid Q(3) \} \\ &= \text{prime}'(2) \wedge \text{prime}'(3) \\ &= [\![2 \text{ is prime}]\!]_t \wedge [\![3 \text{ is prime}]\!]_t \\ &= [\![2 \text{ is prime and } 3 \text{ is prime}]\!] \end{aligned}$$

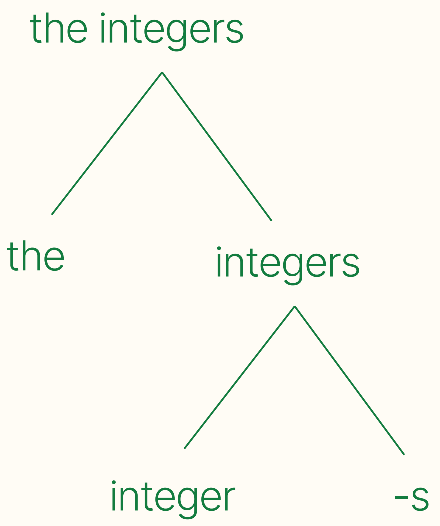
Flexible!

Partee, B., Rooth, M.: Generalized conjunction and type ambiguity. In: Bäuerle, R., Schwarze, C., von Stechow, A. (eds.) Meaning, Use, and Interpretation of Language, pp. 361–383. De Gruyter, Berlin, Boston (1983)

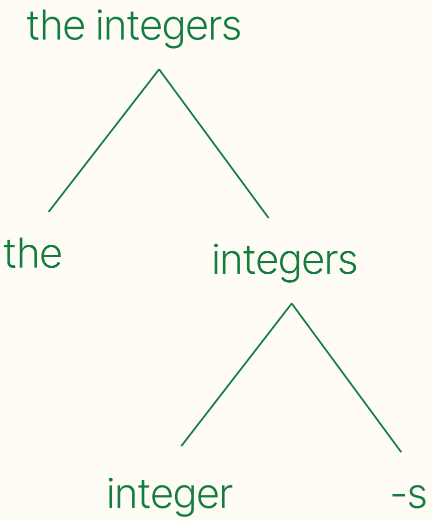
Sets as Subjects

The integers are countable

- [[-s]]
- [[the]]
- [[integer]]
- [[countable]]



Plural Forming *-s*



$$[\![\text{integer}]\!]_{et} = Z$$

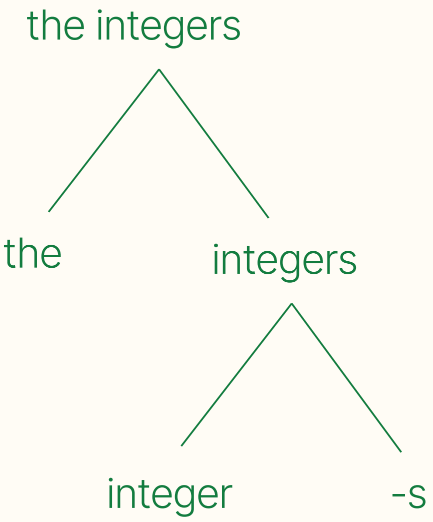
$$[\![\text{-s}]\!]_{(et)(et)t} = \lambda P_{et}\, \lambda Q_{et}(Q \neq \varnothing \wedge Q \subseteq P)$$

$$[\![\text{integers}]\!]_{(et)t} = [\![\text{-s}]\!]\, [\![\text{integer}]\!]$$
$$= \lambda Q_{et}(Q \neq \varnothing \wedge Q \subseteq Z)$$

Link, G.: The logical analysis of plurals and mass terms: A lattice-theoretical approach. In: Bäuerle, R., Schwarze, C., von Stechow, A. (eds.) Meaning, Use, and Interpretation of Language, pp. 302–323. De Gruyter, Berlin, Boston (1983).

Sharvy, R.: A more general theory of definite descriptions. The Philosophical Review 89(4), 607–624 (1980),

Plural Definite Article



$[\![\text{the}]\!]_{((et)t)et} = \lambda P_{(et)t}\,(\text{unique largest } A_{et} \text{ s.t. } P(A))$

$[\![\text{integers}]\!]_{(et)t} = \lambda Q_{et}(Q \neq \varnothing \wedge Q \subseteq Z)$

$[\![\text{the integers}]\!]_{et} = [\![\text{the}]\!]\ [\![\text{integers}]\!] = Z$

$[\![\text{the reals}]\!]_{et} = R$

Sharvy, R.: A more general theory of definite descriptions. The Philosophical Review 89(4), 607–624 (1980)

Sets as Subjects: Revisited

countable : (et)t

countable′(Z), countable′(Q)	True
countable′(R)	False

$[\![\text{the integers are countable}]\!]$

$[\![\text{countable}]\!]_{(et)t}\ [\![\text{the integers}]\!]_{et}$

$= \text{countable}'(Z)$

$[\![\text{the reals are countable}]\!]$

$[\![\text{countable}]\!]_{(et)t}\ [\![\text{the reals}]\!]_{et}$

$= \text{countable}'(R)$

Where Conventional Typing Goes Wrong

Both sets and properties have type et

⟦The integers and the reals⟧
= ⟦the integers⟧$_{et}$ ∩ ⟦the reals⟧$_{et}$
= Z ∩ R
= Z

⟦The integers and the reals are countable⟧
= countable′(Z)
= ⟦The integers are countable⟧

⟦prime and even⟧
= ⟦prime⟧$_{et}$ ∩ ⟦even⟧$_{et}$

- x is prime and even
- x belongs to the intersection of prime integers and even integers
- x is prime and x is even
- prime′(x) ∧ even′(x)

Winter’s Solution and its Problems

Winter's Solution

$[\![\text{the}]\!]_{((et)t)(et)t} = \lambda P_{(et)t}.\ \{\text{unique largest } A_{et} \text{ s.t. } P(A)\}$

$[\![\text{the integers}]\!]_{(et)t} = [\![\text{the}]\!]\ [\![\text{integers}]\!]$

$= \{\ Z\ \}$

countable : $(et)t$, $[\![\text{the integers}]\!]$: $(et)t$

$$\boxed{\mathbf{E}_{(\sigma t)(\sigma t)t} = \lambda A_{\sigma t}\ \lambda B_{\sigma t}\ \exists X_\sigma\ (A(X) \wedge B(X))}$$

$[\![\text{the integers are countable}]\!]$

$= \mathbf{E}(\text{countable}')([\![\text{the integers}]\!])$

$= \mathbf{E}(\text{countable}')(\{Z\})$

$= \exists X_{et}(\text{countable}(X) \wedge X \in \{Z\})$

$= \text{countable}'(Z)$

Winter, Y.: Flexibility Principles in Boolean Semantics: The Interpretation of Coordination, Plurality, and Scope in Natural Language, chap. Coordination and Collectivity. The MIT Press (02 2002).

Partee, B.H.: Noun phrase interpretation and type-shifting principles. In: J. Groenendijk, D. de Jongh, M. Stokhof (eds.) Studies in Discourse Representation Theory and the Theory of Generalized Quantifiers, pp. 115–144. De Gruyter, Berlin, Boston (1986)

Problem with Winter's Solution

the integers and the reals

the integers   the reals

1. $\mathbf{E}(\llbracket \text{the integers} \rrbracket \cap \llbracket \text{the reals} \rrbracket)(\text{countable}')$
   $= \exists X_{et}(X \in \{Z\} \cap \{R\}) \wedge \text{countable}'(X))$
   $= \exists X_{et}(X \in \varnothing) \wedge \text{countable}'(X))$

2. $(\mathbf{E}\llbracket \text{the integers} \rrbracket) \cap (\mathbf{E}\llbracket \text{the reals} \rrbracket))(\text{countable}')$
   $= \text{countable}'(Z) \wedge \text{countable}'(R)$

A Typing Fix

A Typing Fix

Core issue: Sets and properties behave differently when coordinated

Fix: 1. Change the type system by adding a unary constructor S

e	t	S(e)	et	S(e)t	(et)t	S(S(e))t	…

2. Ensure “the integers” and “the reals” have type S(e) instead of et.

⟦-s⟧ : (et)S(e)t and **not** ⟦-s⟧ : (et)(et)t
⟦the⟧ : (S(e)t)S(e) and **not** ⟦the⟧ : ((et)t)et

3. Do **not** define “and” as ∩ for S(e)

Typing Fix and Flexibility for Correct Derivations

Fix: 4. Extend **M**

$$\mathbf{M} = \lambda x_e.\{ P_{et} \mid P(x) \}$$
$$\mathbf{M'} = \lambda A_{S(e)}.\{ P_{S(e)t} \mid P(A) \}$$

$$\begin{aligned}[\![\text{The integers and the reals}]\!]_{(S(e)t)t} &= \mathbf{M'}([\![\text{the integers}]\!]) \cap \mathbf{M'}([\![\text{the reals}]\!]) \\ &= \mathbf{M'}(Z) \cap \mathbf{M'}(R) \\ &= \{P_{S(e)t} \mid P(Z)\} \cap \{Q_{S(e)t} \mid Q(R)\}\end{aligned}$$

$[\![\text{The integers and the reals are countable}]\!]$

$= [\![\text{The integers and the reals}]\!]_{(S(e)t)t}\ [\![\text{countable}]\!]_{S(e)t}$

$= \text{countable}' \in \{P_{S(e)t} \mid P(Z)\} \cap \{Q_{S(e)t} \mid Q(R)\}$

$= \text{countable}'(Z) \wedge \text{countable}'(R)$

$= [\![\text{The integers are countable and the reals are countable}]\!]$

Conclusion, Limitations and Future Work

- Flexible Boolean Semantics
- Coordination of sets differs from coordination of properties
- Typing fix which distinguishes between sets and properties

"The integers and reals are countable" is true.

- Quantificational vs predicative NPs
- Atom (*be numerous*) vs Set Predicates (*gather*)
- Generalized Quantifier Theory
    - All primes more than y are odd.
    - 6 has exactly m factors.

ON IT

Winter, Yoad. Flexibility principles in Boolean semantics: The interpretation of coordination, plurality, and scope in natural language. MIT press, 2002.

Dowty, David. "Collective predicates, distributive predicates, and all." Proceedings of the 3rd ESCOL. Ohio: Ohio State University, 1987.