Some observations about plurals in textual mathematics

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Based on:

- unpublished joint work with Santiago Arambillete;
- discussions with Aarne Ranta, Hugo Herbelin, Paul-André Melliès, and Yoad Winter.

Overview

- Textual Mathematics
- Compositional Semantics
- Plurals
- Collective vs Distributive Predicates
- Symmetric Predicates
- Conclusions

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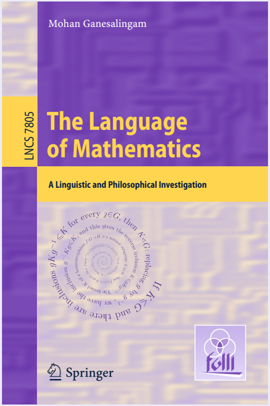
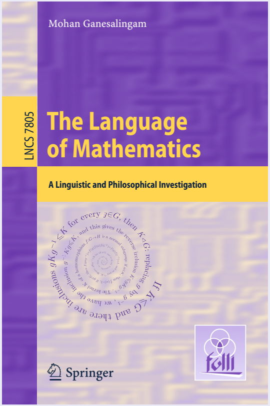
Textual Mathematics



Textual mathematics:

- the language used by mathematicians in textbooks and articles;
- consists of a mixture of natural language and mathematical formulas;
- it has its own idiosyncrasies that are worth studying.

Textual Mathematics



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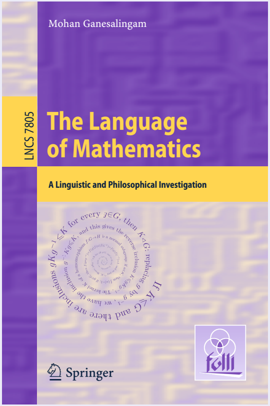

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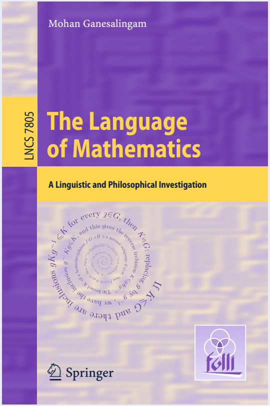

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- consists of a mixture of natural language and mathematical formulas;
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Textual Mathematics

Linguistically, the study of mathematical language rather than everyday language is rewarding because it offers examples that have complicated grammatical structure but are free from ambiguities. We always know exactly what a sentence means, and there is a determinate structure to be revealed. The informal language of mathematics thus provides a kind of grammatical laboratory.

Ranta (1994)

Textual Mathematics

Soit f une forme *hermitienne* sur un espace vectoriel L sur K. On dit que deux vecteurs $x, y \in \mathrm{L}$ sont **orthogonaux** par rapport à f si

$$f(x, y) = 0.$$

(...)

Soit maintenant M un sous-espace vectoriel de L; on appelle **orthogonal de** M par rapport à f l'ensemble, noté généralement

$$\mathrm{M}^{\perp},$$

des $x \in \mathrm{L}$ qui sont orthogonaux à *tout* $y \in \mathrm{M}$.

Godement, *Cours d'Algèbre*

Textual Mathematics

Let f be a *hermitian* form on a vector space L over K. Two vectors $x, y \in \mathrm{L}$ are said to be **orthogonal** with respect to f if

$$f(x, y) = 0.$$

(...)

Now let M be a vector subspace of L. The **orthogonal complement of** M with respect to f is defined to be the set, usually denoted by

$$\mathrm{M}^{\perp},$$

of all vectors $x \in \mathrm{L}$ which are orthogonal to *every* $y \in \mathrm{M}$.

Compositional semantics

Abstract Syntactic Structures

A farmer feeds a gray donkey.

$$\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\,(\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))$$

$$\begin{aligned}
\text{FARMER} &: \text{N} \\
\text{DONKEY} &: \text{N} \\
\text{GRAY} &: \text{N} \rightarrow \text{N} \\
\text{FEED} &: \text{NP} \rightarrow \text{NP} \rightarrow \text{S} \\
\text{SOME} &: \text{N} \rightarrow \text{QNP} \\
\text{QR} &: \text{QNP} \rightarrow (\text{NP} \rightarrow \text{S}) \rightarrow \text{S}
\end{aligned}$$

Compositional semantics

Abstract Syntactic Structures

A farmer feeds a gray donkey.

$$\text{QR}\,(\text{SOME FARMER})\,(\lambda x.\,\text{QR}\,(\text{SOME}\,(\text{GRAY DONKEY}))\,(\lambda y.\,\text{FEED}\,y\,x))$$

$$\begin{aligned}
\text{FARMER} &: \text{N}\\
\text{DONKEY} &: \text{N}\\
\text{GRAY} &: \text{N} \to \text{N}\\
\text{FEED} &: \text{NP} \to \text{NP} \to \text{S}\\
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Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
[\![\mathrm{N}]\!] &= \mathbf{e} \rightarrow \mathbf{t} \\
[\![\mathrm{NP}]\!] &= \mathbf{e} \\
[\![\mathrm{QNP}]\!] &= (\mathbf{e} \rightarrow \mathbf{t}) \rightarrow \mathbf{t} \\
[\![\mathrm{S}]\!] &= \mathbf{t}
\end{aligned}
$$

$$
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[\![\mathrm{FARMER}]\!] &= \lambda x.\, \textbf{farmer}\, x \\
[\![\mathrm{DONKEY}]\!] &= \lambda x.\, \textbf{donkey}\, x \\
[\![\mathrm{GRAY}]\!] &= \lambda p x.\, (p\, x) \wedge (\textbf{gray}\, x) \\
[\![\mathrm{FEED}]\!] &= \lambda x y.\, \textbf{feed}\, y\, x \\
[\![\mathrm{SOME}]\!] &= \lambda p q.\, \exists x.\, (p\, x) \wedge (q\, x) \\
[\![\mathrm{QR}]\!] &= \lambda f x.\, f\, x
\end{aligned}
$$

where $\textbf{farmer}, \textbf{donkey}, \textbf{gray} : \mathrm{e} \rightarrow \mathrm{t}$
$\textbf{feed} : \mathrm{e} \rightarrow \mathrm{e} \rightarrow \mathrm{t}$

Compositional semantics

Semantic Interpretation

$$\begin{aligned} [\![\mathrm{N}]\!] &= \mathbf{e} \to \mathbf{t} \\ [\![\mathrm{NP}]\!] &= \mathbf{e} \\ [\![\mathrm{QNP}]\!] &= (\mathbf{e} \to \mathbf{t}) \to \mathbf{t} \\ [\![\mathrm{S}]\!] &= \mathbf{t} \end{aligned}$$

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Compositional semantics

Semantic Interpretation

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\begin{aligned}
&[\![\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
&\quad = [\![\textsc{qr}]\!]\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\quad = (\lambda f x.\,f\,x)\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\quad = (\lambda pq.\,\exists x.\,(p\,x) \wedge (q\,x))\,(\lambda x.\,\textbf{farmer}\,x) \\
&\qquad\qquad\qquad\qquad (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge [\![(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge ((\lambda f x.\,f\,x)\,[\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge ([\![\textsc{some}\ (\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge ((\lambda pq.\,\exists y.\,(p\,y) \wedge (q\,y))\,[\![(\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,([\![\textsc{gray donkey}]\!]\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
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Compositional semantics

Semantic Interpretation

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&\llbracket \textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))\rrbracket \\
&\quad = \llbracket \textsc{qr}\rrbracket\ \llbracket(\textsc{some farmer})\rrbracket\ (\lambda x.\,\llbracket \textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)\rrbracket) \\
&\quad = (\lambda f x.\,f\,x)\ \llbracket(\textsc{some farmer})\rrbracket\ (\lambda x.\,\llbracket \textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)\rrbracket) \\
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&\quad = (\lambda pq.\,\exists x.\,(p\,x) \wedge (q\,x))\ (\lambda x.\,\mathbf{farmer}\,x) \\
&\qquad\qquad\qquad (\lambda x.\,\llbracket \textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)\rrbracket) \\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge \llbracket(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))\rrbracket \\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda f x.\,f\,x)\ \llbracket(\textsc{some}\ (\textsc{gray donkey}))\rrbracket\ (\lambda y.\,\llbracket\textsc{feed}\rrbracket\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\llbracket\textsc{some}\ (\textsc{gray donkey})\rrbracket\ (\lambda y.\,\llbracket\textsc{feed}\rrbracket\,y\,x)) \\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda pq.\,\exists y.\,(p\,y) \wedge (q\,y))\ \llbracket(\textsc{gray donkey})\rrbracket\ (\lambda y.\,\llbracket\textsc{feed}\rrbracket\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\llbracket\textsc{gray donkey}\rrbracket\,y) \wedge (\llbracket\textsc{feed}\rrbracket\,y\,x)) \\
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\end{aligned}
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Compositional semantics

Semantic Interpretation

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Compositional semantics

Semantic Interpretation

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\end{aligned}
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Compositional semantics

Semantic Interpretation

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&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge ((\lambda x y.\,\textbf{feed}\,y\,x)\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge (\textbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\text{QR}\,(\text{SOME FARMER})\,(\lambda x.\,\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x))]\!]\\
&\quad = [\![\text{QR}]\!]\;[\![(\text{SOME FARMER})]\!]\;(\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\quad = (\lambda f x.\,f\,x)\;[\![(\text{SOME FARMER})]\!]\;(\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta [\![\text{SOME FARMER}]\!]\;(\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\quad = (\lambda pq.\,\exists x.\,(p\,x) \wedge (q\,x))\;(\lambda x.\,\mathbf{farmer}\,x)\\
&\qquad\qquad\qquad\qquad (\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge [\![(\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x))]\!]\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda f x.\,f\,x)\;[\![(\text{SOME (GRAY DONKEY)})]\!]\;(\lambda y.\,[\![\text{FEED}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge ([\![\text{SOME (GRAY DONKEY)}]\!]\;(\lambda y.\,[\![\text{FEED}]\!]\,y\,x))\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda pq.\,\exists y.\,(p\,y) \wedge (q\,y))\;[\![(\text{GRAY DONKEY})]\!]\;(\lambda y.\,[\![\text{FEED}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,([\![\text{GRAY DONKEY}]\!]\,y) \wedge ([\![\text{FEED}]\!]\,y\,x))\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,((\lambda px.\,(p\,x) \wedge (\mathbf{gray}\,x))\,(\lambda x.\,\mathbf{donkey}\,x)\,y) \wedge ([\![\text{FEED}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge ([\![\text{FEED}]\!]\,y\,x))\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge ((\lambda xy.\,\mathbf{feed}\,y\,x)\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge (\mathbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad = [\![\textsc{qr}]\!]\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad = (\lambda f x.\,f\,x)\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad = (\lambda pq.\,\exists x.\,(p\,x) \wedge (q\,x))\,(\lambda x.\,\mathbf{farmer}\,x)\\
&\qquad\qquad\qquad\qquad (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge [\![(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda f x.\,f\,x)\,[\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge ([\![\textsc{some}\ (\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda pq.\,\exists y.\,(p\,y) \wedge (q\,y))\,[\![(\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,([\![\textsc{gray donkey}]\!]\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,((\lambda p x.\,(p\,x) \wedge (\mathbf{gray}\,x))\,(\lambda x.\,\mathbf{donkey}\,x)\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge ((\lambda xy.\,\mathbf{feed}\,y\,x)\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge (\mathbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
&\quad = [\![\textsc{qr}]\!]\ [\![(\textsc{some farmer})]\!]\ (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\quad = (\lambda f x.\,f\ x)\ [\![(\textsc{some farmer})]\!]\ (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\ (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\quad = (\lambda pq.\,\exists x.\,(p\,x) \wedge (q\,x))\ (\lambda x.\,\textbf{farmer}\,x) \\
&\qquad\qquad\qquad\qquad (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge [\![(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge ((\lambda f x.\,f\ x)\ [\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\ (\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge ([\![\textsc{some}\ (\textsc{gray donkey})]\!]\ (\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge ((\lambda pq.\,\exists y.\,(p\,y) \wedge (q\,y))\ [\![(\textsc{gray donkey})]\!]\ (\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,([\![\textsc{gray donkey}]\!]\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,((\lambda p x.\,(p\,x) \wedge (\textbf{gray}\,x))\,(\lambda x.\,\textbf{donkey}\,x)\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge ((\lambda xy.\,\textbf{feed}\,y\,x)\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge (\textbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{array}{l}
[\![\mathrm{QR}\,(\textsc{some farmer})\,(\lambda x.\,\mathrm{QR}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
\quad = [\![\mathrm{QR}]\!]\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\mathrm{QR}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
\quad = (\lambda f x.\,f\,x)\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\mathrm{QR}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\,(\lambda x.\,[\![\mathrm{QR}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
\quad = (\lambda pq.\,\exists x.\,(p\,x) \wedge (q\,x))\,(\lambda x.\,\mathbf{farmer}\,x) \\
\qquad\qquad\qquad\qquad (\lambda x.\,[\![\mathrm{QR}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge [\![(\mathrm{QR}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda f x.\,f\,x)\,[\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge ([\![\textsc{some}\ (\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge ((\lambda pq.\,\exists y.\,(p\,y) \wedge (q\,y))\,[\![(\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,([\![\textsc{gray donkey}]\!]\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,((\lambda p x.\,(p\,x) \wedge (\mathbf{gray}\,x))\,(\lambda x.\,\mathbf{donkey}\,x)\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge ([\![\textsc{feed}]\!]\,y\,x)) \\
\quad = \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge ((\lambda xy.\,\mathbf{feed}\,y\,x)\,y\,x)) \\
\twoheadrightarrow_\beta \exists x.\,(\mathbf{farmer}\,x) \wedge (\exists y.\,(\mathbf{donkey}\,y) \wedge (\mathbf{gray}\,y) \wedge (\mathbf{feed}\,x\,y))
\end{array}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad= [\![\textsc{qr}]\!]\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad= (\lambda f x.\,f\,x)\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad= (\lambda pq.\,\exists x.\,(p\,x)\wedge(q\,x))\,(\lambda x.\,\textbf{farmer}\,x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge[\![(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad= \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda f x.\,f\,x)\,[\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge([\![\textsc{some}\ (\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\quad= \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda pq.\,\exists y.\,(p\,y)\wedge(q\,y))\,[\![(\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,([\![\textsc{gray donkey}]\!]\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\quad= \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,((\lambda p x.\,(p\,x)\wedge(\textbf{gray}\,x))\,(\lambda x.\,\textbf{donkey}\,x)\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\quad= \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge((\lambda xy.\,\textbf{feed}\,y\,x)\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge(\textbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad = [\![\textsc{qr}]\!]\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad = (\lambda f x.\,f\,x)\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad = (\lambda pq.\,\exists x.\,(p\,x)\wedge(q\,x))\,(\lambda x.\,\textbf{farmer}\,x)\\
&\qquad\qquad\qquad\qquad\qquad (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge[\![(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda f x.\,f\,x)\,[\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge([\![\textsc{some}\ (\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda pq.\,\exists y.\,(p\,y)\wedge(q\,y))\,[\![(\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,([\![\textsc{gray donkey}]\!]\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,((\lambda p x.\,(p\,x)\wedge(\textbf{gray}\,x))\,(\lambda x.\,\textbf{donkey}\,x)\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge((\lambda xy.\,\textbf{feed}\,y\,x)\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge(\textbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\text{QR}\,(\text{SOME FARMER})\,(\lambda x.\,\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x))]\!]\\
&\quad = [\![\text{QR}]\!]\;[\![(\text{SOME FARMER})]\!]\,(\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\quad = (\lambda f x.\,f\,x)\;[\![(\text{SOME FARMER})]\!]\,(\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta [\![\text{SOME FARMER}]\!]\,(\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\quad = (\lambda pq.\,\exists x.\,(p\,x) \wedge (q\,x))\,(\lambda x.\,\textbf{farmer}\,x)\\
&\qquad\qquad\qquad\qquad (\lambda x.\,[\![\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge [\![(\text{QR}\,(\text{SOME (GRAY DONKEY)})\,(\lambda y.\,\text{FEED}\,y\,x))]\!]\\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge ((\lambda f x.\,f\,x)\,[\![(\text{SOME (GRAY DONKEY)})]\!]\,(\lambda y.\,[\![\text{FEED}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge ([\![\text{SOME (GRAY DONKEY)}]\!]\,(\lambda y.\,[\![\text{FEED}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge ((\lambda pq.\,\exists y.\,(p\,y) \wedge (q\,y))\,[\![(\text{GRAY DONKEY})]\!]\,(\lambda y.\,[\![\text{FEED}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,([\![\text{GRAY DONKEY}]\!]\,y) \wedge ([\![\text{FEED}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,((\lambda p x.\,(p\,x) \wedge (\textbf{gray}\,x))\,(\lambda x.\,\textbf{donkey}\,x)\,y) \wedge ([\![\text{FEED}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge ([\![\text{FEED}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge ((\lambda x y.\,\textbf{feed}\,y\,x)\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x) \wedge (\exists y.\,(\textbf{donkey}\,y) \wedge (\textbf{gray}\,y) \wedge (\textbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad = [\![\textsc{qr}]\!]\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad = (\lambda f x.\,f\,x)\,[\![(\textsc{some farmer})]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\,(\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\quad = (\lambda pq.\,\exists x.\,(p\,x)\wedge(q\,x))\,(\lambda x.\,\textbf{farmer}\,x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!])\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge[\![(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!]\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda f x.\,f\,x)\,[\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge([\![\textsc{some}\ (\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda pq.\,\exists y.\,(p\,y)\wedge(q\,y))\,[\![(\textsc{gray donkey})]\!]\,(\lambda y.\,[\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,([\![\textsc{gray donkey}]\!]\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,((\lambda p x.\,(p\,x)\wedge(\textbf{gray}\,x))\,(\lambda x.\,\textbf{donkey}\,x)\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge([\![\textsc{feed}]\!]\,y\,x))\\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge((\lambda xy.\,\textbf{feed}\,y\,x)\,y\,x))\\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge(\textbf{feed}\,x\,y))
\end{aligned}
$$

Compositional semantics

Semantic Interpretation

$$
\begin{aligned}
&[\![\textsc{qr}\,(\textsc{some farmer})\,(\lambda x.\,\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
&\quad = [\![\textsc{qr}]\!]\ [\![(\textsc{some farmer})]\!]\ (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\quad = (\lambda f x.\,f\,x)\ [\![(\textsc{some farmer})]\!]\ (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\twoheadrightarrow_\beta [\![\textsc{some farmer}]\!]\ (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\quad = (\lambda pq.\,\exists x.\,(p\,x)\wedge(q\,x))\ (\lambda x.\,\textbf{farmer}\,x) \\
&\qquad\qquad\qquad\qquad (\lambda x.\,[\![\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x)]\!]) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge[\![(\textsc{qr}\,(\textsc{some}\ (\textsc{gray donkey}))\,(\lambda y.\,\textsc{feed}\,y\,x))]\!] \\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda f x.\,f\,x)\ [\![(\textsc{some}\ (\textsc{gray donkey}))]\!]\ (\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge([\![\textsc{some}\ (\textsc{gray donkey})]\!]\ (\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge((\lambda pq.\,\exists y.\,(p\,y)\wedge(q\,y))\ [\![(\textsc{gray donkey})]\!]\ (\lambda y.\,[\![\textsc{feed}]\!]\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,([\![\textsc{gray donkey}]\!]\,y)\wedge([\![\textsc{feed}]\!]\,y\,x)) \\
&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,((\lambda p x.\,(p\,x)\wedge(\textbf{gray}\,x))\,(\lambda x.\,\textbf{donkey}\,x)\,y)\wedge([\![\textsc{feed}]\!]\,y\,x)) \\
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&\quad = \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge((\lambda xy.\,\textbf{feed}\,y\,x)\,y\,x)) \\
&\twoheadrightarrow_\beta \exists x.\,(\textbf{farmer}\,x)\wedge(\exists y.\,(\textbf{donkey}\,y)\wedge(\textbf{gray}\,y)\wedge(\textbf{feed}\,x\,y))
\end{aligned}
$$

Plurals

Three approaches

- Mereology: $a \oplus b$. (Blunt, 1985)
- Plural logic: $\forall x, \forall xs$. (Nicolas, 2008)
- Second-order logic: plural as sets of entities, i.e, terms of type $\mathrm{e} \rightarrow \mathrm{t}$. (Link, 1983)

Plural logic can be formalized as a fragment of second order-logic.

Mereology = Boolean algebra without 0.

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Syntactic categories

$$[\![\mathrm{N_{sg}}]\!] = \mathbf{e} \rightarrow \mathbf{t} \qquad [\![\mathrm{N_{pl}}]\!] = (\mathbf{e} \rightarrow \mathbf{t}) \rightarrow \mathbf{t}$$

$$[\![\mathrm{NP_{sg}}]\!] = \mathbf{e} \qquad [\![\mathrm{NP_{pl}}]\!] = \mathbf{e} \rightarrow \mathbf{t}$$

$$[\![\mathrm{QNP_{sg}}]\!] = (\mathbf{e} \rightarrow \mathbf{t}) \rightarrow \mathbf{t} \qquad [\![\mathrm{QNP_{pl}}]\!] = ((\mathbf{e} \rightarrow \mathbf{t}) \rightarrow \mathbf{t}) \rightarrow \mathbf{t}$$

Plurals

Link's distributivity operator

$$\mathbf{distr} \triangleq \lambda pS. \forall x. (S\, x) \rightarrow (p\, x)$$

Plurals

A farmer feeds some donkeys.

$$
\begin{aligned}
\text{FARMER} &: \mathrm{N}_{\mathrm{sg}} \\
\text{DONKEY} &: \mathrm{N}_{\mathrm{sg}} \\
\text{FEED} &: \mathrm{NP}_{\mathrm{sg}} \to \mathrm{NP}_{\mathrm{sg}} \to \mathrm{S} \\
\text{SOME}_{\mathrm{sg}} &: \mathrm{N}_{\mathrm{sg}} \to \mathrm{QNP}_{\mathrm{sg}} \\
\text{SOME}_{\mathrm{pl}} &: \mathrm{N}_{\mathrm{pl}} \to \mathrm{QNP}_{\mathrm{pl}} \\
\text{QR}_{\mathrm{sg}} &: \mathrm{QNP}_{\mathrm{sg}} \to (\mathrm{NP}_{\mathrm{sg}} \to \mathrm{S}) \to \mathrm{S} \\
\text{QR}_{\mathrm{pl}} &: \mathrm{QNP}_{\mathrm{pl}} \to (\mathrm{NP}_{\mathrm{pl}} \to \mathrm{S}) \to \mathrm{S} \\
\text{PL} &: \mathrm{N}_{\mathrm{sg}} \to \mathrm{N}_{\mathrm{pl}} \\
\text{DISTR} &: (\mathrm{NP}_{\mathrm{sg}} \to \mathrm{S}) \to \mathrm{NP}_{\mathrm{pl}} \to \mathrm{S}
\end{aligned}
$$

$$
\text{QR}_{\mathrm{sg}}\,(\text{SOME}_{\mathrm{sg}}\ \text{FARMER})\,(\lambda x.\,\text{QR}_{\mathrm{pl}}\,(\text{SOME}_{\mathrm{pl}}\ (\text{PL DONKEY}))\,(\text{DISTR}\,(\lambda y.\,\text{FEED}\,y\,x)))
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$$
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$$

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[\![\mathrm{DISTR}]\!] &= \mathbf{distr}
\end{aligned}
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Plurals

THEOREM 2. Let M be a finite-dimensional vector space over a division ring, let X be a finite set of generators of M, and let A be a subset of X. Suppose that the elements of A are linearly independent. Then there exists a basis B of M such that

$$\mathrm{A} \subset \mathrm{B} \subset \mathrm{X}.$$

Collective vs Distributive Predicates

Definition and Examples

A collective predicate, as opposed to a distributive one, is a predicate that applies to a plural entity considered as a whole, rather than to each individual that comprises it.

The soldiers <u>surrounded</u> the fort.

* *Each soldier surrounded the fort.*

The soldiers were <u>numerous</u>.

* *Each soldier was numerous.*

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Collective vs Distributive Predicates

Mathematical Examples

There exists a finite set of elements $a_1, \ldots, a_n \in \mathrm{M}$ which generate M.

Let A be a set of coprime numbers.

Suppose that the elements of A are linearly independent.

Let X be a finite set of generators of M

$$\mathbf{prime} \triangleq \lambda a.\,(a \neq 1) \wedge (\forall n.\,((\mathsf{Nat}\,n) \wedge (\mathbf{div}\,n\,a)) \to ((n = 1) \vee (n = a)))$$
$$\mathbf{coprime} \triangleq \lambda S.\,\forall n.\,((\mathsf{Nat}\,n) \wedge (\forall a.\,(S\,a) \to (\mathbf{div}\,n\,a))) \to (n = 1)$$

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Collective vs Distributive Predicates

Group Nouns

The bataillon surrounded the fort.

Each bataillon surrounded the fort.

It is enough to show that B *generates* M *(since by construction* B *is free).*

Reification

$$\textbf{set} : \mathbf{e} \to (\mathbf{e} \to \mathbf{t}) \to \mathbf{t}$$

$$\mathsf{Nat}\,, \mathsf{Rat}\ : \mathbf{e} \to \mathbf{t}$$
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$$\textbf{set}\, \mathbb{N}\, (\lambda x.\, \mathsf{Nat}\, x)$$
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$$\forall s. \forall S S'. \, ((\textbf{set}\, s\, S) \wedge (\textbf{set}\, s\, S')) \to (S = S')$$

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Collective vs Distributive Predicates

Every set of natural numbers is a set of rational numbers.

$$\begin{aligned} \text{NATURAL-NUMBER} &: \mathrm{N}_{\mathrm{sg}} \\ \text{RATIONAL-NUMBER} &: \mathrm{N}_{\mathrm{sg}} \\ \text{SET-OF} &: \mathrm{N}_{\mathrm{pl}} \to \mathrm{N}_{\mathrm{sg}} \end{aligned}$$

$$\begin{aligned} \mathrm{QR}_{\mathrm{sg}}\ & (\text{EVERY}\,(\text{SET-OF}\,(\text{PL NATURAL-NUMBER}))) \\ & (\lambda x.\, \mathrm{QR}_{\mathrm{sg}}\ (\text{SOME}_{\mathrm{sg}}\,(\text{SET-OF}\,(\text{PL RATIONAL-NUMBER}))) \\ & \qquad (\lambda y.\, x = y)) \end{aligned}$$

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$$\begin{array}{l} \mathrm{QR}_{\mathrm{sg}}\ (\text{EVERY}\,(\text{SET-OF}\,(\text{PL NATURAL-NUMBER}))) \\ \qquad (\lambda x.\, \mathrm{QR}_{\mathrm{sg}}\ (\text{SOME}_{\mathrm{sg}}\,(\text{SET-OF}\,(\text{PL RATIONAL-NUMBER}))) \\ \qquad\qquad (\lambda y.\, x = y)) \end{array}$$

Collective vs Distributive Predicates

Every set of natural numbers is a set of rational numbers.

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\end{aligned}
$$

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\end{aligned}
$$

$$
\begin{array}{ll}
\textsc{qr}_{\,\mathrm{sg}} & (\textsc{every}\,(\textsc{set-of}\,(\textsc{pl}\ \textsc{natural-number})))\\
& (\lambda x.\, \textsc{qr}_{\,\mathrm{sg}}\ \ (\textsc{some}_{\,\mathrm{sg}}\,(\textsc{set-of}\,(\textsc{pl}\ \textsc{rational-number})))\\
& \qquad\qquad\quad (\lambda y.\, x = y))
\end{array}
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Collective vs Distributive Predicates

$$\begin{aligned}
[\![\text{NATURAL-NUMBER}]\!] &= \lambda x.\, \mathsf{Nat}\, x \\
[\![\text{RATIONAL-NUMBER}]\!] &= \lambda x.\, \mathsf{Rat}\, x \\
[\![\text{SET-OF}]\!] &= \lambda P x.\, \exists S.\, (P\, S) \wedge (\textbf{set}\, x\, S)
\end{aligned}$$

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&\forall x.\, (\exists S.\, (\forall z.\, (S\, z) \to (\mathsf{Nat}\, z)) \wedge (\textbf{set}\, x\, S)) \\
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Every set of prime numbers is a set of coprime numbers.

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Symmetric Predicates

Phrases that denote binary symmetric predicates may often be used as collective predicates:

- *James agrees with Carol.*
  Carol agrees with James.
  James and Carol agree.

- *Boston is quite different from New York.*
  New York is quite different from Boston.
  Boston and New York are quite different.

- *Sue and Dan divorced.*
  ?*Sue divorced Dan.*
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Symmetric Predicates

General Scheme

- *A is orthogonal to B.*
  B is orthogonal to A.
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(1) $[\mathrm{NP}_1]$ is $[\mathrm{ADJ}]$ $[\mathrm{PREP}]$ $[\mathrm{NP}_2]$.

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Symmetric Predicates

The vector y is orthogonal to the vector x.

Every row of $\mathbf{H}_X$ is orthogonal to every row of $\mathbf{H}_Z$.

The section s is orthogonal to the first m eigenfunctions of the operator.

The vector y and the vector x are orthogonal.

$^?$Every row of $\mathbf{H}_X$ and every row of $\mathbf{H}_Z$ are orthogonal.

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Symmetric Predicates

A plea for the dual grammatical number

$$\mathrm{AND} : \mathrm{NP}_{\mathrm{sg}} \to \mathrm{NP}_{\mathrm{sg}} \to \mathrm{NP}_{\mathrm{du}}$$

$$\begin{aligned} [\![\mathrm{NP}_{\mathrm{du}}]\!] &= (\mathbf{e} \to \mathbf{e} \to \mathbf{t}) \to \mathbf{t} \\ [\![\mathrm{AND}]\!] &= \lambda xyf.\, f\, x\, y \end{aligned}$$

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Symmetric Predicates

Much more to say:

- Adjectives denoting (symmetric) binary relations used with noun phrases that denote collections of three or more elements.
- Binary distributivity operator.
- Strict binary distributivity operator.
- Overt markers of (strict) binary distributivity: *pairwize, mutually, by pairs...*
- Reciprocals: *each other, one another...*
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Conclusions

- Most of the semantic phenomena that appear in natural language also arise in textual mathematics.
- However, the use of natural language in mathematics tends to be more regular.
- The language of mathematics provides indeed a quite interesting *grammatical laboratory.*
- Studying the linguistic structure of textual mathematics is an interesting source of inspiration for how to formalize mathematicst.

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