A Generic Deskolemization Strategy

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First-Order Logic for Proof Assistants

- sequent calculi or natural deduction
- Here Sequent Calculus GS3 (left-sided)

$$\frac{}{\Gamma, \bot}\ \bot \qquad \frac{}{\Gamma, P, \neg P}\ \text{ax} \qquad \frac{}{\Gamma, \neg\top}\ \neg\top$$

$$\frac{\Gamma, P, Q}{\Gamma, P \land Q}\ \land \qquad \frac{\Gamma, P}{\Gamma, \neg\neg P}\ \neg\neg \qquad \frac{\Gamma, \neg P, \neg Q}{\Gamma, \neg(P \lor Q)}\ \neg\lor \qquad \frac{\Gamma, P, \neg Q}{\Gamma, \neg(P \Rightarrow Q)}\ \neg\Rightarrow$$

$$\frac{\Gamma, P \qquad \Gamma, Q}{\Gamma, P \lor Q}\ \lor \qquad \frac{\Gamma, \neg P \qquad \Gamma, Q}{\Gamma, P \Rightarrow Q}\ \Rightarrow \qquad \frac{\Gamma, \neg P \qquad \Gamma, \neg Q}{\Gamma, \neg(P \land Q)}\ \neg\land$$

$$\frac{\Gamma, \forall x.\, P, P[x \mapsto t]}{\Gamma, \forall x.\, P}\ \forall \qquad \frac{\Gamma, \neg\exists x.\, P, \neg P[x \mapsto t]}{\Gamma, \neg\exists x.\, P}\ \neg\exists$$

$$\frac{\Gamma, \exists x.\, P, P[x \mapsto c^*]}{\Gamma, \exists x.\, P}\ \exists \qquad \frac{\Gamma, \neg\forall x.\, P, \neg P[x \mapsto c^*]}{\Gamma, \neg\forall x.\, P}\ \neg\forall$$

First-Order Logic for Automated Theorem Proving

Refutation methods, rather than direct proofs:

- tableaux, here (Goéland, concurrent & parallel ATP)
- postpone instantiation of universal variables
  - ⋆ leave Free Variables (aka “Meta”) instead
  - ⋆ instantiate at closing time
  - ⋆ freshness of existential constants c^* under threat
  - ⋆ register the variables the constant **depends on**
  - ⋆ Skolem symbol, Skolem term
- ATPs need **proof certificates**

Our Contribution

Translation from free-variable tableaux to sequent calculus.

Free-Variable Tableaux Calculus

- closure rules:

$$\frac{\bot}{\odot}\,\odot_\bot \qquad \frac{\neg\top}{\odot}\,\odot_{\neg\top} \qquad \frac{P,\neg Q}{\sigma}\,\odot_\sigma, \sigma(P)=\sigma(Q)$$

- α rules (non-branching connectives):

$$\frac{\neg\neg P}{P}\,\alpha_{\neg\neg} \qquad \frac{P \wedge Q}{P, Q}\,\alpha_\wedge \qquad \frac{\neg(P \vee Q)}{\neg P, \neg Q}\,\alpha_{\neg\vee} \qquad \frac{\neg(P \Rightarrow Q)}{P, \neg Q}\,\alpha_{\neg\Rightarrow}$$

- β rules (branching connectives):

$$\frac{P \vee Q}{P \quad\quad Q}\,\beta_\vee \qquad \frac{P \Rightarrow Q}{\neg P \quad\quad Q}\,\beta_\Rightarrow \qquad \frac{\neg(P \wedge Q)}{\neg P \quad\quad \neg Q}\,\beta_{\neg\wedge}$$

- γ rules (universal quantifiers):

$$\frac{\forall x.\, P}{P[x \mapsto X]}\,\gamma_\forall \qquad \frac{\neg\exists x.\, P}{\neg P[x \mapsto X]}\,\gamma_{\neg\exists}$$

- δ rules (existential quantifiers):

$$\frac{\exists x.\, P}{P[x \mapsto f(X_1, \ldots, X_n)]}\,\delta_\exists \qquad \frac{\neg\forall x.\, P}{\neg P[x \mapsto f(X_1, \ldots, X_n)]}\,\delta_{\neg\forall}$$

Example Tableau Proof: Drinker's Principle

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y))}{\neg(D(X) \Rightarrow \forall y\, D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\, D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\{X \mapsto c\}}\,\odot_{\sigma}$$

Tableau Proof

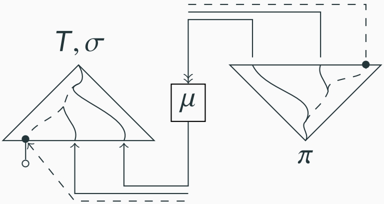
Proof tree and a uniform substitution σ that closes all the branches at once.

From Tableaux to Sequent Calculus

- usual induction : impossible (why : in a minute)
- instead : grow the sequent proof from the root
- needs discipline : maintain a mapping

Mapping

a function μ that associates to each leave of a sequent proof tree, an (internal) node of the tableau proof.



- the node that a leaves maps to is “the next rule to be incorporated”.

Extraction of the Drinker's Principle Tableau Proof

$$
\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

Extraction of the Drinker's Principle Tableau Proof

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$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash$

Extraction of the Drinker's Principle Tableau Proof

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$$\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\exists}$$

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$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\exists}$$

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Let the fun begin !

$$\cfrac{\cfrac{\overline{\overline{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\exists}$$

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$$

$$
\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\exists}
$$

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$$\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\exists}$$

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$$
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$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg_\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_\exists
$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

Replay first, to grow back the missing formulas

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\exists$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\exists}$$

Extraction of the Drinker's Principle Tableau Proof

$$
\begin{array}{c|l}
\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)) & \\
\hline
\neg(D(X) \Rightarrow \forall y\, D(y)) & \gamma_{\neg\exists} \\
\hline
D(X), \neg(\forall y\, D(y)) & \alpha_{\neg\Rightarrow} \\
\hline
\neg D(c) & \delta^{+}_{\neg\forall} \\
\hline
\sigma = \{X \mapsto c\} & \odot_{\sigma}
\end{array}
$$

$$
\begin{array}{c|l}
\hline
\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(\forall y\, D(y)), \neg D(c) \vdash & \neg\exists \\
\hline
\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(\forall y\, D(y)) \vdash & \neg\forall \\
\hline\hline
\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(D(c) \Rightarrow \forall y\, D(y)), D(c), \neg(\forall y\, D(y)) \vdash & \mathrm{W} \times 2 \\
\hline
\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(D(c) \Rightarrow \forall y\, D(y)) \vdash & \neg\Rightarrow \\
\hline
\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)) \vdash & \neg\exists
\end{array}
$$

Extraction of the Drinker's Principle Tableau Proof

$$\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\dfrac{\neg(D(X) \Rightarrow \forall y\ D(y))}{\dfrac{D(X), \neg(\forall y\ D(y))}{\dfrac{\neg D(c)}{\sigma = \{X \mapsto c\}}\ \odot_\sigma}\ \delta^+_{\neg\forall}}\ \alpha_{\neg\Rightarrow}}\ \gamma_{\neg\exists}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_\exists$$

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Extraction of the Drinker's Principle Tableau Proof

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(D(c) \Rightarrow \forall y\, D(y)), D(c), \neg(\forall y\, D(y)), \neg D(c) \vdash}\,\text{ax}}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(D(c) \Rightarrow \forall y\, D(y)), \neg(\forall y\, D(y)), \neg D(c) \vdash}\,\neg_{\Rightarrow}}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(\forall y\, D(y)), \neg D(c) \vdash}\,\neg_{\exists}}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(\forall y\, D(y)) \vdash}\,\neg_{\forall}}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(D(c) \Rightarrow \forall y\, D(y)), D(c), \neg(\forall y\, D(y)) \vdash}\,\mathsf{W} \times 2}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \neg(D(c) \Rightarrow \forall y\, D(y)) \vdash}\,\neg_{\Rightarrow}}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)) \vdash}\,\neg_{\exists}
$$

Key Steps

Consider a seq. cal. proof-tree π, and a mapping to a tableau T.

- Given a tableau non δ-rule to be incorporated : do it.
- If it is a δ-rule :
  1. weaken the formulas containing the offensive Skolem ($\simeq$ fresh) term.
  2. apply the δ-rule (no more offense here)
  3. regenerate all the weakened formulas by replaying rules of π.
- note : the mapping is lost after weakening. We must regain it after step 3. This is hard.

Basic Notions

Context : tableau proof T with substitution σ.

Descendance

The descendance of F on a branch of T is the sequence of formulas originating from F.

Dependency

$\forall x.A$ depends on $\exists y.D$ iff one of its direct descendant uses the Skolem term.

Dependency descendance

Sequence of descendants of a dependent F, that contain the Skolem term of interest and that appear on the node.

Consider the following tableau proof

$$\cfrac{\cfrac{\forall y.((P(y) \wedge \exists x.\neg P(x)) \vee F)}{(P(Y) \wedge \exists x.\neg P(x)) \vee F}\,\gamma_\forall}{\cfrac{\cfrac{\cfrac{P(Y) \wedge \exists x.\neg P(x)}{P(Y), \exists x.\neg P(x)}\,\alpha_\wedge}{\neg P(c)}\,\delta_\exists}{\{Y \mapsto c\}}\,\odot \qquad \cfrac{F}{\vdots}}\,\beta_\vee$$

A Hydra Game

π : proof-tree (under construction) for the F-branch.

$$\dfrac{\dfrac{\dfrac{\psi, \cdots, P(c), \exists x. \neg P(x) \vdash}{\psi, \cdots, P(c) \wedge (\exists x. \neg P(x)) \vdash} \wedge \qquad \pi}{\psi, \cdots, (P(c) \wedge (\exists x. \neg P(x))) \vee F \vdash} \vee}{\psi = \forall y. ((P(y) \wedge (\exists x. \neg P(x))) \vee F) \vdash} \forall$$

A Hydra Game

π : proof-tree (under construction) for the F-branch.

Step 1 : weaken* the offensive formulas (c-dependent descendants)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\psi, \exists x.\, \neg P(x) \vdash}{\psi, \cdots, P(c), \exists x.\, \neg P(x) \vdash}\ \mathrm{W} \times 3}{\psi, \cdots, P(c) \wedge (\exists x.\, \neg P(x)) \vdash}\ \wedge^{*} \qquad \pi}{\psi, \cdots, (P(c) \wedge (\exists x.\, \neg P(x))) \vee F \vdash}\ \vee^{*}}{\psi = \forall y.\, ((P(y) \wedge (\exists x.\, \neg P(x))) \vee F) \vdash}\ \forall^{*}
$$

A Hydra Game

π : proof-tree (under construction) for the F-branch.

Step 2 : $\exists$ rule and Skolem term/fresh symbol

$$
\dfrac{\dfrac{\dfrac{\dfrac{\psi, \exists x.\neg P(x), \neg P(c) \vdash}{\psi, \boxed{\exists x.\neg P(x)} \vdash}\ \boxed{\exists}}{\psi, \cdots, P(c), \exists x.\neg P(x) \vdash}\ \mathrm{W}\times 3}{\psi, \cdots, P(c) \wedge (\exists x.\neg P(x)) \vdash}\ \wedge^* \qquad \pi}{\dfrac{\psi, \cdots, (P(c) \wedge (\exists x.\neg P(x))) \vee F \vdash}{\psi = \forall y.((P(y) \wedge (\exists x.\neg P(x))) \vee F) \vdash}\ \forall^*}\ \vee^*
$$

A Hydra Game

π : proof-tree (under construction) for the F-branch.

Step 3 : replay the gray* rules, grow back weakened formulas

$$
\begin{prooftree}
\AxiomC{$\cdots, \neg P(c), \cdots, (P(c) \wedge (\exists x.\, \neg P(x))) \vee F \vdash$}
\RightLabel{$\boxed{\forall}$}
\UnaryInfC{$\boxed{\psi}, \exists x.\, \neg P(x), \neg P(c) \vdash$}
\RightLabel{$\exists$}
\UnaryInfC{$\psi, \exists x.\, \neg P(x) \vdash$}
\doubleLine
\RightLabel{$\mathrm{W} \times 3$}
\UnaryInfC{$\psi, \cdots, P(c), \exists x.\, \neg P(x) \vdash$}
\RightLabel{$\wedge^*$}
\UnaryInfC{$\psi, \cdots, P(c) \wedge (\exists x.\, \neg P(x)) \vdash$}
\AxiomC{π}
\RightLabel{$\vee^*$}
\BinaryInfC{$\psi, \cdots, (P(c) \wedge (\exists x.\, \neg P(x))) \vee F \vdash$}
\RightLabel{$\boxed{\forall^*}$}
\UnaryInfC{$\psi = \forall y.\, ((P(y) \wedge (\exists x.\, \neg P(x))) \vee F) \vdash$}
\end{prooftree}
$$

A Hydra Game

π : proof-tree (under construction) for the F-branch.

Step 3 : **Beware of branching rules**

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              \cdots,\neg P(c),\cdots,P(c)\wedge(\exists x.\,\neg P(x))\vdash \qquad \pi
            }{
              \cdots,\neg P(c),\cdots,(P(c)\wedge(\exists x.\,\neg P(x)))\vee F\vdash
            }\vee
          }{
            \psi,\exists x.\,\neg P(x),\neg P(c)\vdash
          }\forall
        }{
          \psi,\exists x.\,\neg P(x)\vdash
        }\exists
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        \psi,\cdots,P(c),\exists x.\,\neg P(x)\vdash
      }\mathrm{W}\times 3
    }{
      \psi,\cdots,P(c)\wedge(\exists x.\,\neg P(x))\vdash
    }\wedge^{*}
    \qquad \pi
  }{
    \psi,\cdots,(P(c)\wedge(\exists x.\,\neg P(x)))\vee F\vdash
  }\vee^{*}
}{
  \psi=\forall y.\,((P(y)\wedge(\exists x.\,\neg P(x)))\vee F)\vdash
}\forall^{*}
$$

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π : proof-tree (under construction) for the F-branch.

Step 3 : replay the gray* rules, grow back weakened formulas

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cdots,\neg P(c),\cdots,P(c),\exists x.\neg P(x)\vdash}{\cdots,\neg P(c),\cdots,P(c)\wedge(\exists x.\neg P(x))\vdash}\,\wedge \qquad \pi}{\cdots,\neg P(c),\cdots,(P(c)\wedge(\exists x.\neg P(x)))\vee F\vdash}\,\vee}{\psi,\exists x.\neg P(x),\neg P(c)\vdash}\,\forall}{\psi,\exists x.\neg P(x)\vdash}\,\exists}{\psi,\cdots,P(c),\exists x.\neg P(x)\vdash}\,\mathrm{W}\times 3}{\psi,\cdots,P(c)\wedge(\exists x.\neg P(x))\vdash}\,\wedge^{*} \qquad \pi}{\psi,\cdots,(P(c)\wedge(\exists x.\neg P(x)))\vee F\vdash}\,\vee^{*}}{\psi=\forall y.((P(y)\wedge(\exists x.\neg P(x)))\vee F)\vdash}\,\forall^{*}
$$

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π : proof-tree (under construction) for the F-branch.

Then : keep going (**maybe !**)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots, \boxed{\neg P(c)}, \cdots, \boxed{P(c)}, \exists x.\, \neg P(x) \vdash}\,\mathrm{ax}}{\cdots, \neg P(c), \cdots, P(c) \wedge (\exists x.\, \neg P(x)) \vdash}\,\wedge \qquad \pi}{\cdots, \neg P(c), \cdots, (P(c) \wedge (\exists x.\, \neg P(x))) \vee F \vdash}\,\vee}{\psi, \exists x.\, \neg P(x), \neg P(c) \vdash}\,\forall}{\psi, \exists x.\, \neg P(x) \vdash}\,\exists$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\psi, \exists x.\, \neg P(x) \vdash}{\psi, \cdots, P(c), \exists x.\, \neg P(x) \vdash}\,\mathrm{W} \times 3}{\psi, \cdots, P(c) \wedge (\exists x.\, \neg P(x)) \vdash}\,\wedge^* \qquad \pi}{\psi, \cdots, (P(c) \wedge (\exists x.\, \neg P(x))) \vee F \vdash}\,\vee^*}{\psi = \forall y.\, ((P(y) \wedge (\exists x.\, \neg P(x))) \vee F) \vdash}\,\forall^*$$

The Crux of the Problem

$$\exists^* \, \cfrac{\cfrac{\vdots \qquad \pi}{\textit{(replay)}}}{\cfrac{\vdots \qquad \pi}{G \vdash}}$$

An annoying chain of consequences:

- replay missing branching rule $\Rightarrow$ duplicate π
- different sequent leaves mapped to the *same* tableau rule
- some leaves of π may be mapped to the $\exists/\delta$-rule to incorporate
- incoporating a rule on **one** leaf $\Rightarrow$ more hydra heads
- this happens (not on the example, though)
- does not look like we are making any progress
- **takeaway: Step 3 (replay) is complex**

Meet the Strategy Rules

Strategy Rule

When replaying the $\vee$, if a branch of π maps to the same δ-rule, additionally keep **relevant** formulas.

- we provide a list of **conditions** that strategy rules must satisfy
- we prove that those conditions ensure termination
- and we also prove

Proposition

It is ok to do so (will not break any other freshness condition in π).

- works for any skolemization that respects some **requirements**
- at least : outer, inner, pre-inner.

More Detail

- **Skolem term** $\approx$ **constant**, as soon as rules are in the correct order. So we get a real sequent calculus proof.
- **Pain is mandatory** : Skolemization leads to huge speed-up. Deskolemizing cannot avoid huge blow-up (at places).
- **Ensuring progression** : a strategy rule has to yield a smaller mapping at the end.

Benchmark

	Problems Proved	Avg. proof size	Avg. size increase	Max. size increase	Avg. time deskolem-ization (ms)	Avg. time translation Seq. Cal. $\rightarrow$ Coq (ms)
Goéland	261	6.9	0 %	—	72.1	15.5
Goéland+δ^+	272	7.0	8.1 %	$\times$ 5.3	75.8	14.4
Goéland+δ^{+^+}	274	7.1	10.6 %	$\times$ 10.3	134.1	39.3
Goéland +DMT	363	6.4	0 %	—	63.4	11.1
Goéland +DMT+δ^+	375	6.5	4.5 %	$\times$ 3.9	72.1	12.1
Goéland +DMT+δ^{+^+}	377	6.5	7.4 %	$\times$ 5.2	76.1	12.1

- proofs by Goéland (parallel & concurrent tableaux ATP)
- addition of Deduction Modulo Theory (DMT)
- translated into Coq
- size increase, deskolemization time : reasonable

Conclusion[1]

What have we seen?

- a strategy to replace Skolem symbols by fresh constants
- in first-order classical logic
- modular in the Skolemization chosen (outer, inner, pre-inner)
- modular in some critical steps (rule replay)
- works in practice (feat. nastily crafted proofs)

What will we do next?

- extension to Deduction Modulo Theory
- extensions of logic (higher-order, dependent types – Dedukti)
- generalize wrt to the order nodes linked to a same δ-rule are processed (depth-first, parallel)

[1] Thanks to LIRMM, Alma Mater of JR and JC, esp. H. Bouziane, S. Robillard and D. Delahaye.