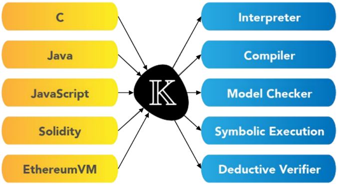
Part 2. The $\mathbb{K}$ framework in Dedukti

Amélie LEDEIN

$\mathbb{K}$ framework in a nutshell

- Semantical framework
  - to define formal semantics of programming languages
  - to automatically generate tools from these semantics



$\mathbb{K}$ framework in a nutshell

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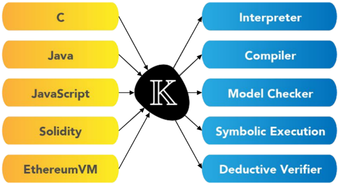
- Based on MATCHING LOGIC

  → an untyped 1st order logic with fixpoints and a "next" operator



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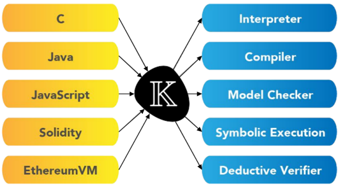


- Based on MATCHING LOGIC
  - → an untyped 1st order logic with fixpoints and a "next" operator

- Common feature: $\mathbb{K}$ and DEDUKTI are based on rewriting.

Characteristic of rewriting	$\mathbb{K}$	DEDUKTI
At any position	✓	✓
Non-linearity	✓	✓
Conditional	✓	✗
Rewriting modulo ACUI	✓	✗

Define a semantics with $\mathbb{K}$

Two steps to define a $\mathbb{K}$ semantics:

- **Syntax**
- **Semantics**

Define a semantics with $\mathbb{K}$

Two steps to define a $\mathbb{K}$ semantics:

- **Syntax**
  - **BNF grammar**
- **Semantics**

Define a semantics with $\mathbb{K}$

Two steps to define a $\mathbb{K}$ semantics:

- **Syntax**
  - **BNF grammar**

- **Semantics**
  - **Configuration** = State of the program
    Example: $\langle\langle\ x + 17\ \rangle_k\ \langle\ x \mapsto 25\ \rangle_{env}\rangle$
  - **Rewriting rule** on configurations ($\sim$ transition system)

Define a semantics with $\mathbb{K}$

$\langle$ `x = 1 ; while 0 < x { x-- } ;` $\rangle_k$
$\langle$ `nil` $\rangle_{env}$

$\langle$ `while 0 < x { x-- } ;` $\rangle_k$
$\langle$ `x` $\mapsto$ `1` $\rangle_{env}$

$\langle$ `while 0 < x { x-- } ;` $\rangle_k$
$\langle$ `x` $\mapsto$ `42` $\rangle_{env}$

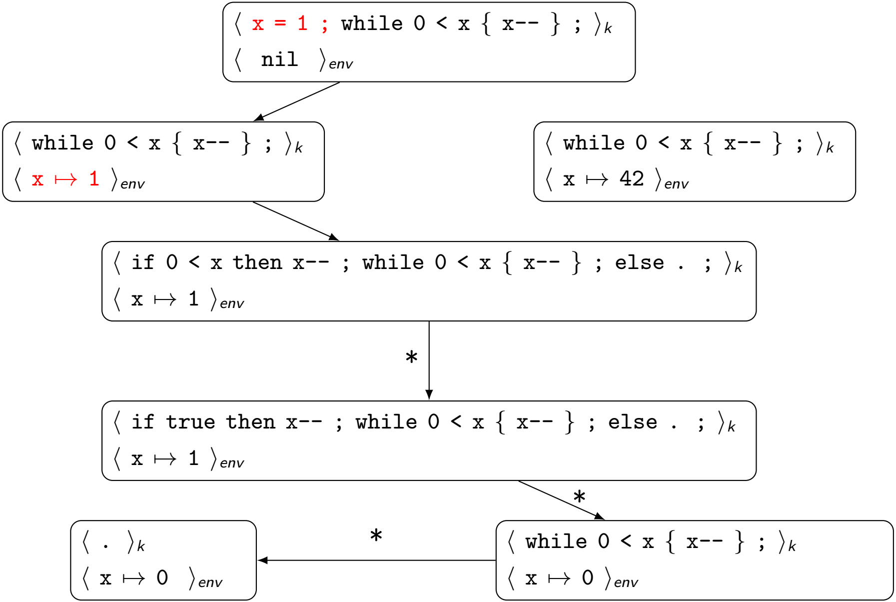
$\langle$ `if 0 < x then x-- ; while 0 < x { x-- } ; else . ;` $\rangle_k$
$\langle$ `x` $\mapsto$ `1` $\rangle_{env}$

$\langle$ `if true then x-- ; while 0 < x { x-- } ; else . ;` $\rangle_k$
$\langle$ `x` $\mapsto$ `1` $\rangle_{env}$

$\langle$ `.` $\rangle_k$
$\langle$ `x` $\mapsto$ `0` $\rangle_{env}$

$\langle$ `while 0 < x { x-- } ;` $\rangle_k$
$\langle$ `x` $\mapsto$ `0` $\rangle_{env}$

Define a semantics with 𝕂



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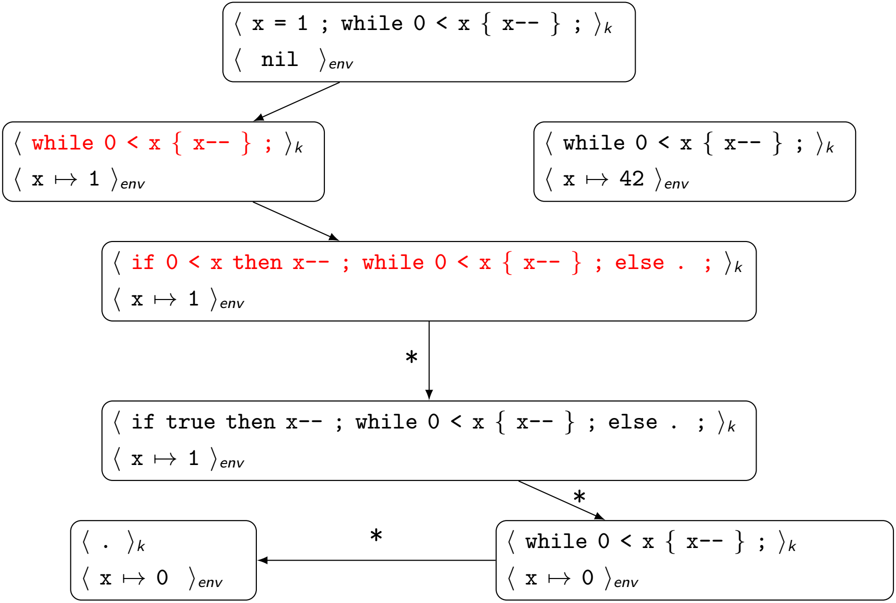

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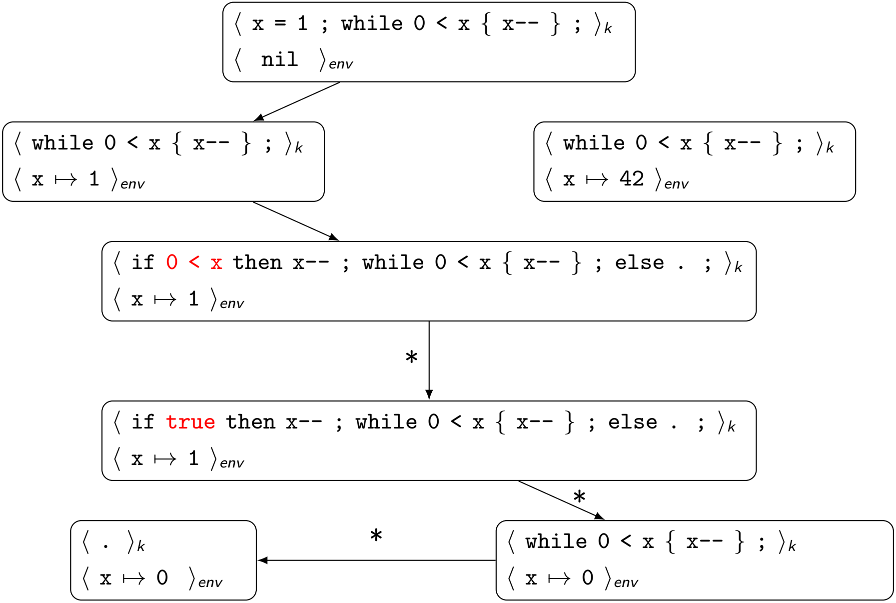

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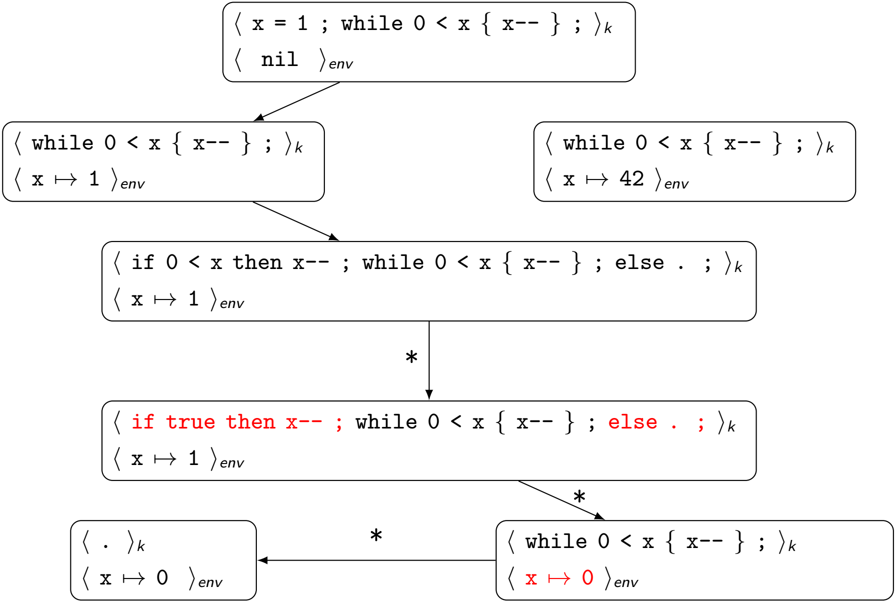

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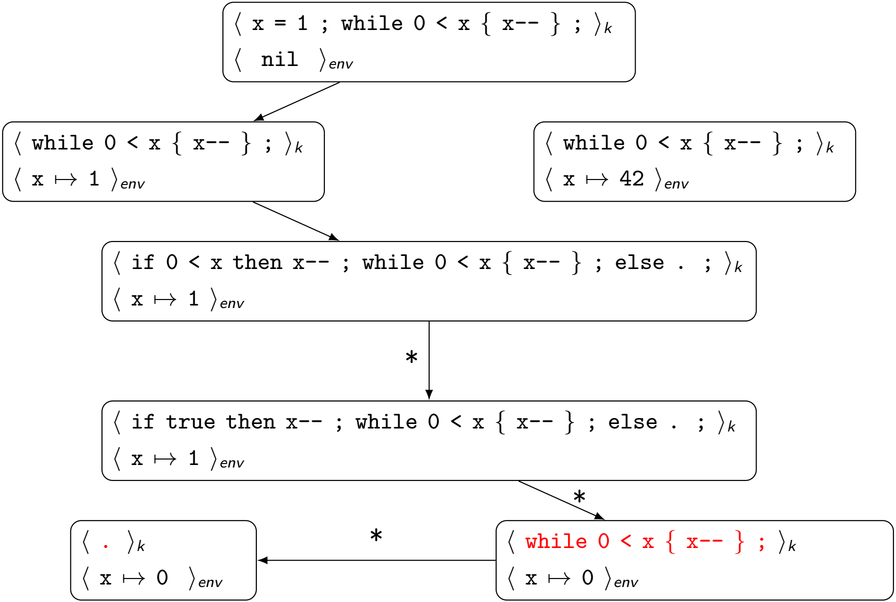
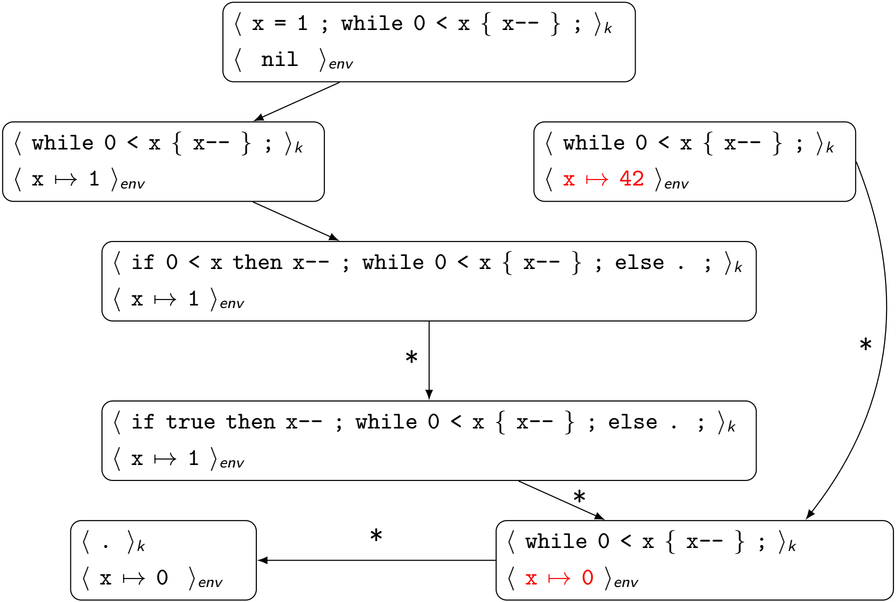
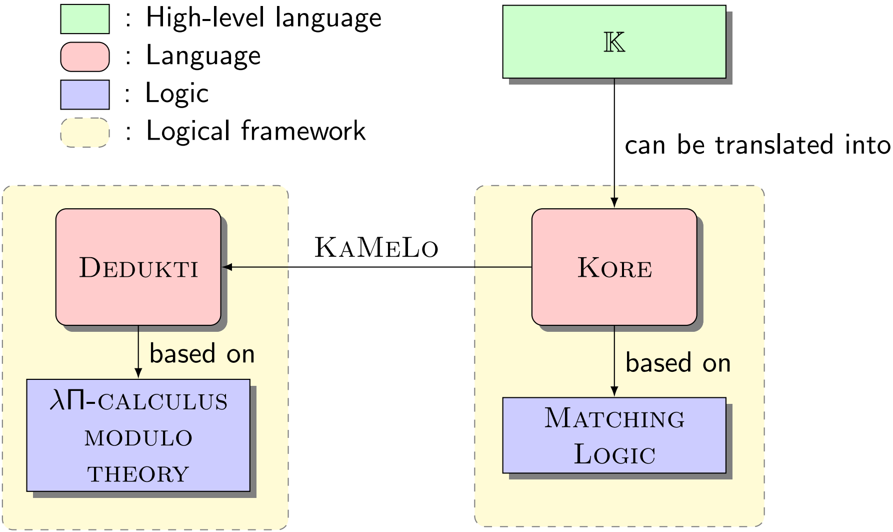


Define a semantics with 𝕂



Pipeline of the translation



Translate $\mathbb{K}$ into DEDUKTI

How to do it?

~~How to do it?~~

- **What is the purpose of the translation?**

Translate $\mathbb{K}$ into DEDUKTI

~~How to do it?~~

- **What is the purpose of the translation?**
- **What do we want to do with the result of the translation?**
  - Execute a program? $\rightsquigarrow$ shallow encoding
  - Check a proof? $\rightsquigarrow$ deep encoding

1. A shallow encoding to execute a program in Dedukti

2. A deep encoding to check proofs in Dedukti
   - Translate Matching Logic constructors, notations and symbols
   - Translate Matching Logic proof system

3. Conclusion

Kore: A Matching Logic theory

```
axiom{R} \exists{R} (Val:SortInt{}, \equals{SortInt{}, R} (Val:SortInt{}, Lbl'UndsPlus'Int'Unds'{}(K0:SortInt{}, K1:SortInt{}))) [functional{}()] // functional
axiom{R} \exists{R} (Val:SortString{}, \equals{SortString{}, R} (Val:SortString{}, Lbl'UndsPlus'String'UndsUnds'STRING-COMMON'Unds'String'Unds'String'Unds'String{}(K0:SortString{}, K1:SortString{}))) [f
axiom{R} \exists{R} (Val:SortAExp{}, \equals{SortAExp{}, R} (Val:SortAExp{}, Lbl'UndsPlusUndsUnds'IMP-SYNTAX'Unds'AExp'Unds'AExp'Unds'AExp{}(K0:SortAExp{}, K1:SortAExp{}))) [functional{}()] // functiona
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```
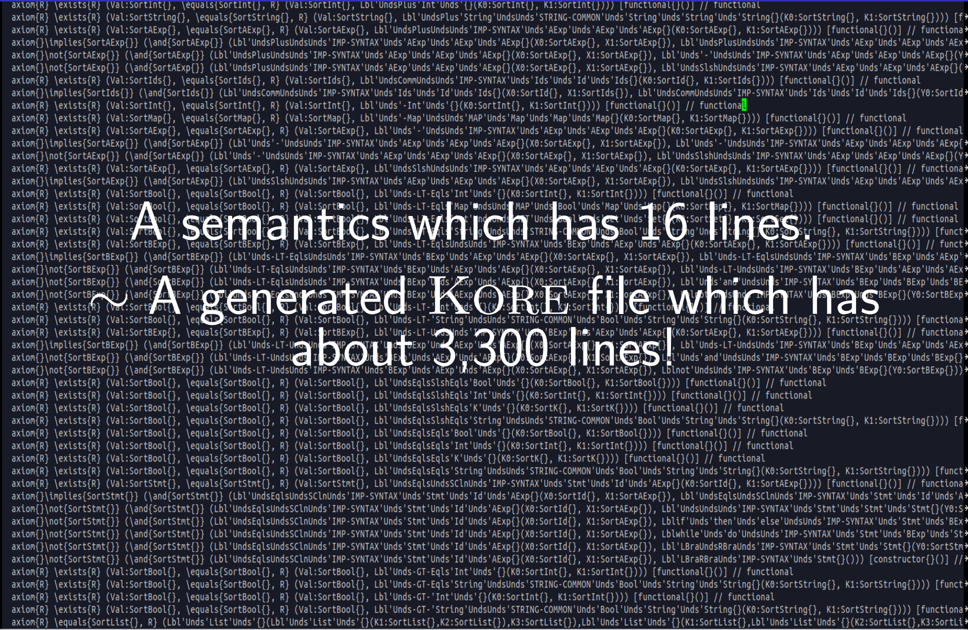
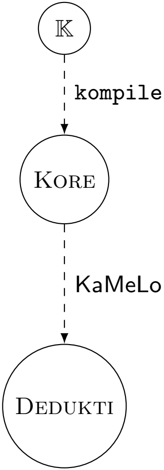

KORE: A MATCHING LOGIC theory



A semantics which has 16 lines.

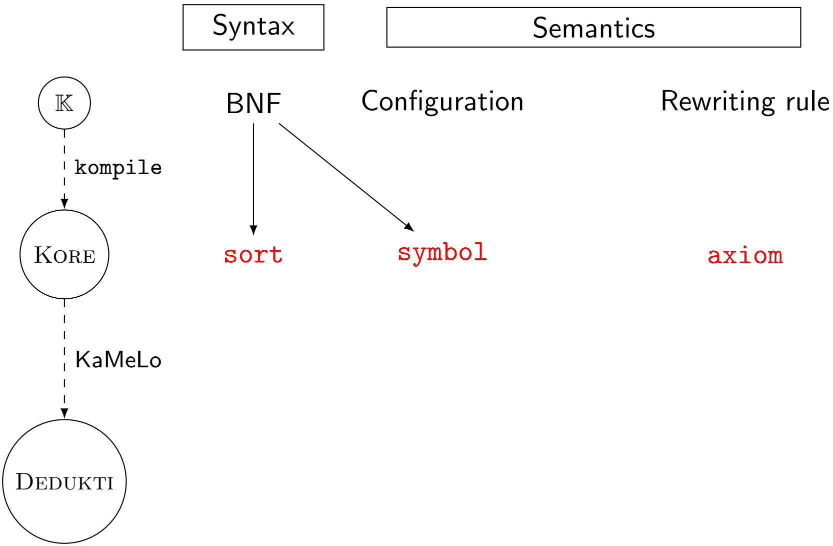
$\sim$ A generated KORE file which has about 3,300 lines!

Translation from $\mathbb{K}$ to DEDUKTI



Syntax	Semantics	
BNF	Configuration	Rewriting rule
`sort`	`symbol`	`axiom`

Translation from $\mathbb{K}$ to DEDUKTI



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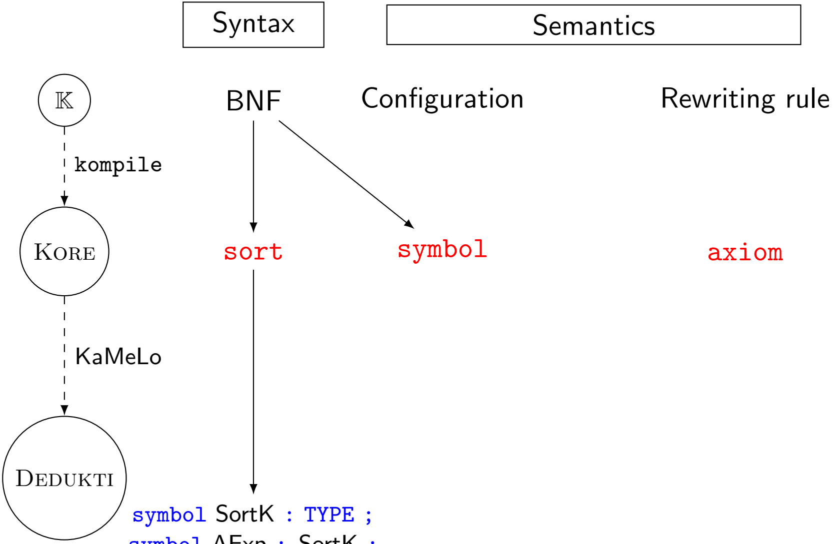
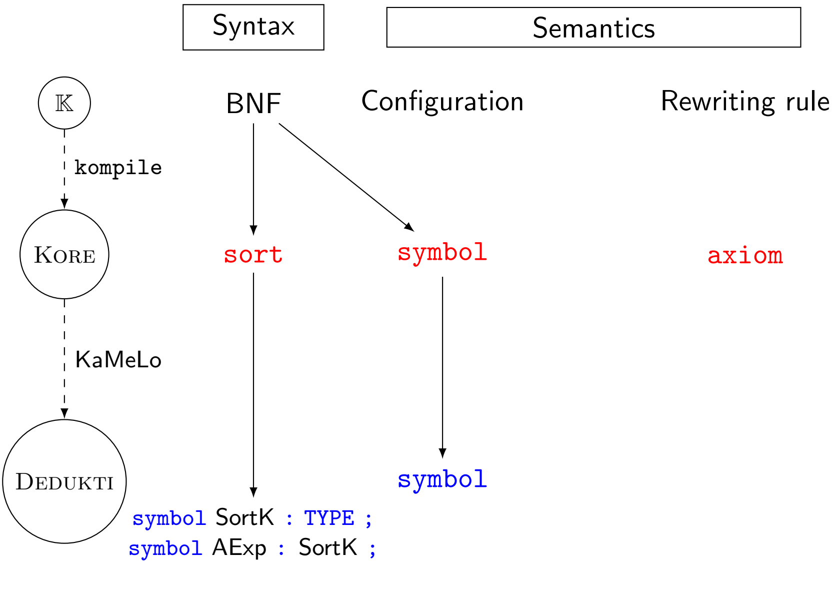



```
symbol SortK : TYPE ;
symbol AExp : SortK ;
```

Translation from $\mathbb{K}$ to DEDUKTI



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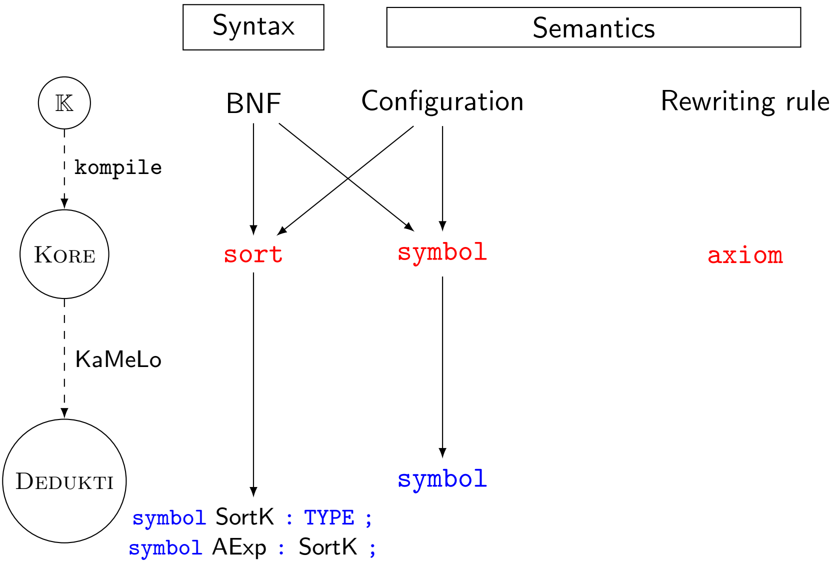
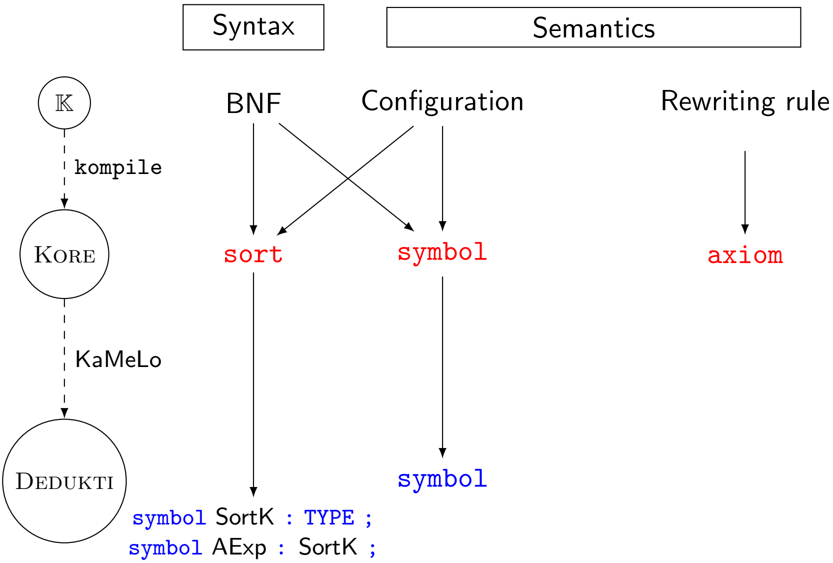


Translation from 𝕂 to DEDUKTI



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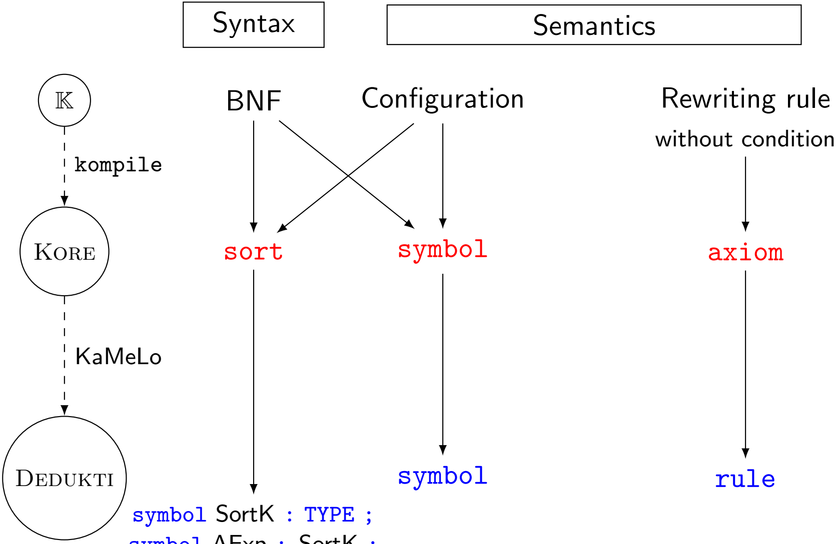


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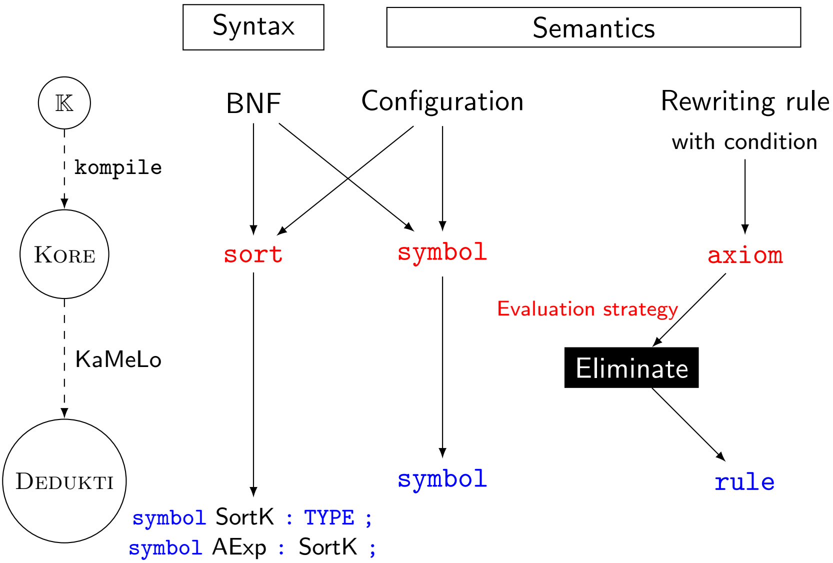

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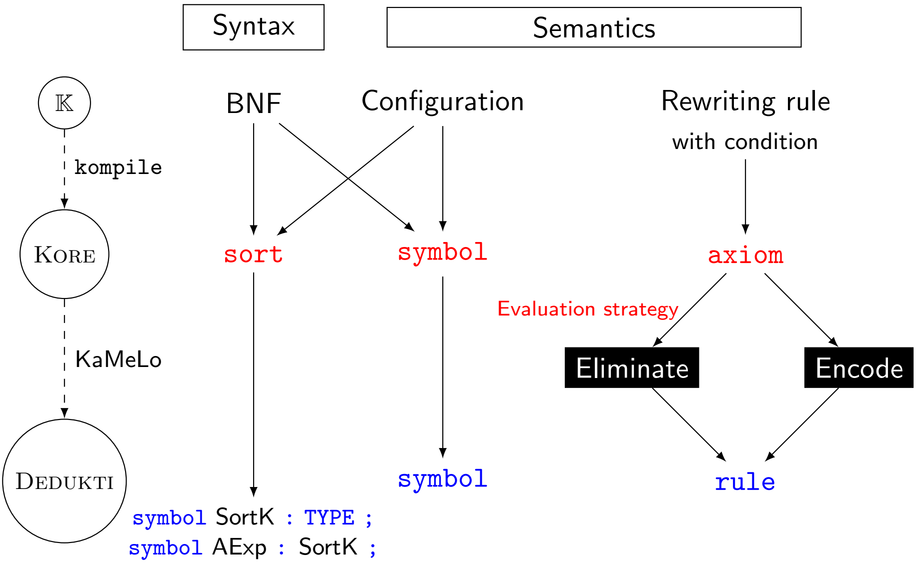

Translate evaluation strategies

Generated rules to define evaluation strategies:

Key ideas:

- **Evaluation is ordering thanks to a list.**

  $(E_1$ `and` $E_2) \curvearrowright E_3 \curvearrowright .$
- **Evaluated expressions have a specific type.**

  `true` and `false` has the type **BExp**

  `true` has the type `Bool`

Translate evaluation strategies

Generated rules to define evaluation strategies:

1. rule E_1 and E_2 => $E_1 \curvearrowright (❄^1_{\text{and}}\ E_2)$ requires $E_1 \notin$ Bool
2. rule $E_1 \curvearrowright (❄^1_{\text{and}}\ E_2)$ => E_1 and E_2 requires $E_1 \in$ Bool

Translation into DEDUKTI:

1. Instantiation of E_1:

   a. rule $\langle\ (\textit{not}\ \$X1)\ \textit{and}\ \$E2 \curvearrowright \$s\ \rangle_k$
   $\hookrightarrow\ \langle\ (\textit{not}\ \$X1) \curvearrowright (❄^1_{\text{and}}\ \$E2) \curvearrowright \$s\ \rangle_k$

   b. rule $\langle\ (\$X1\ \textit{and}\ \$X2)\ \textit{and}\ \$E2 \curvearrowright \$s\ \rangle_k$
   $\hookrightarrow\ \langle\ (\$X1\ \textit{and}\ \$X2) \curvearrowright (❄^1_{\text{and}}\ \$E2) \curvearrowright \$s\ \rangle_k$

2. rule $\langle\ (\textit{inj}\ \$E1) \curvearrowright (❄^1_{\text{and}}\ \$E2) \curvearrowright \$s\ \rangle_k$
   $\hookrightarrow\ \langle\ (\textit{inj}\ \$E1)\ \textit{and}\ \$E2 \curvearrowright \$s\ \rangle_k$

The grammar of **BExp**:

```
syntax BExp ::= Bool
     | "not" BExp
     > BExp "and" BExp
     | "(" BExp ")"
```

Translate a CTRS to a TRS[1]

[1] Patrick Viry, *Elimination of Conditions*, Journal of Symbolic Computation, 1999

Translate a CTRS to a TRS[1]

- **Example 1**:
  (1) `rule` *max* X Y `=>` Y `requires` X `<Int` Y
  (2) `rule` *max* X Y `=>` X `requires` X `>=Int` Y

[1] Patrick Viry, *Elimination of Conditions*, Journal of Symbolic Computation, 1999

Translate a CTRS to a TRS[1]

- **Example 1**:
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  (2) `rule` *max* X Y `=>` X `requires` X `>=Int` Y

  translated into

  (0) `rule` *max* $\$x\ \$y \hookrightarrow \flat max\ \$x\ \$y\ (\$x < \$y)\ (\$x \geq \$y)$

[1] Patrick Viry, *Elimination of Conditions*, Journal of Symbolic Computation, 1999

Translate a CTRS to a TRS[1]

- **Example 1**:

  (1) `rule` *max* X Y `=>` Y `requires` X `<Int` Y
  (2) `rule` *max* X Y `=>` X `requires` X `>=Int` Y

  translated into

  (0) `rule` *max* $\$x\ \$y \hookrightarrow \flat max\ \$x\ \$y\ (\$x < \$y)\ (\$x \geq \$y)$
  (1') `rule` $\flat$*max* $\$x\ \$y$ *true* _ $\hookrightarrow \$y$
  (2') `rule` $\flat$*max* $\$x\ \$y$ _ *true* $\hookrightarrow \$x$

[1] Patrick Viry, *Elimination of Conditions*, Journal of Symbolic Computation, 1999

Translate a CTRS to a TRS[1]

- **Example 1**:

  (1) `rule` *max* X Y `=>` Y `requires` X `<Int` Y
  (2) `rule` *max* X Y `=>` X `requires` X `>=Int` Y

  translated into

  (0) `rule` *max* $x $y $\hookrightarrow$ $\flat$*max* $x $y ($x $<$ $y) ($x $\geq$ $y)
  (1') `rule` $\flat$*max* $x $y *true* _ $\hookrightarrow$ $y
  (2') `rule` $\flat$*max* $x $y _ *true* $\hookrightarrow$ $x

- **Example 2**:

  (A) `rule` *max* X Y `=>` Y `requires` X `<Int` Y
  (B) `rule` *max* X Y `=>` X [`owise`]

  translated into

  (א) `rule` *max* $x $y $\hookrightarrow$ $\flat$*max* $x $y ($x $<$ $y)
  (A') `rule` $\flat$*max* $x $y *true* $\hookrightarrow$ $y
  (B') `rule` $\flat$*max* $x $y *false* $\hookrightarrow$ $x

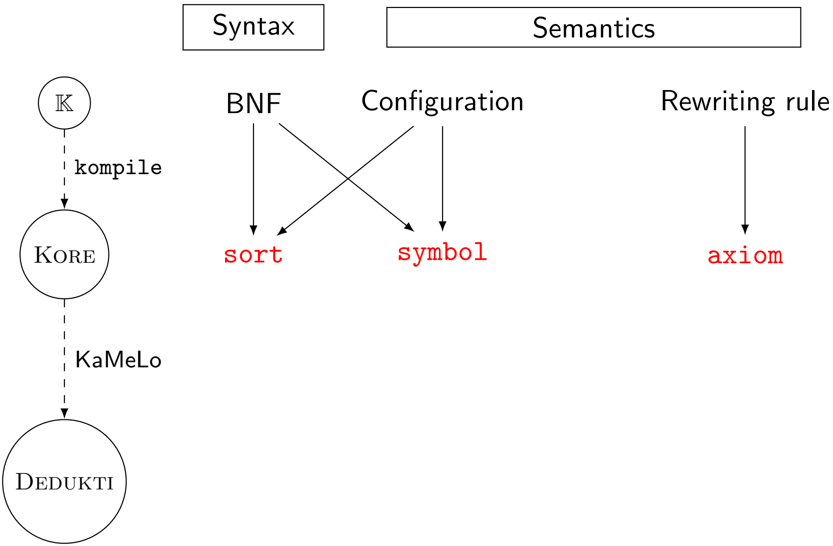
[1]Patrick Viry, *Elimination of Conditions*, Journal of Symbolic Computation, 1999

1. A shallow encoding to execute a program in Dedukti

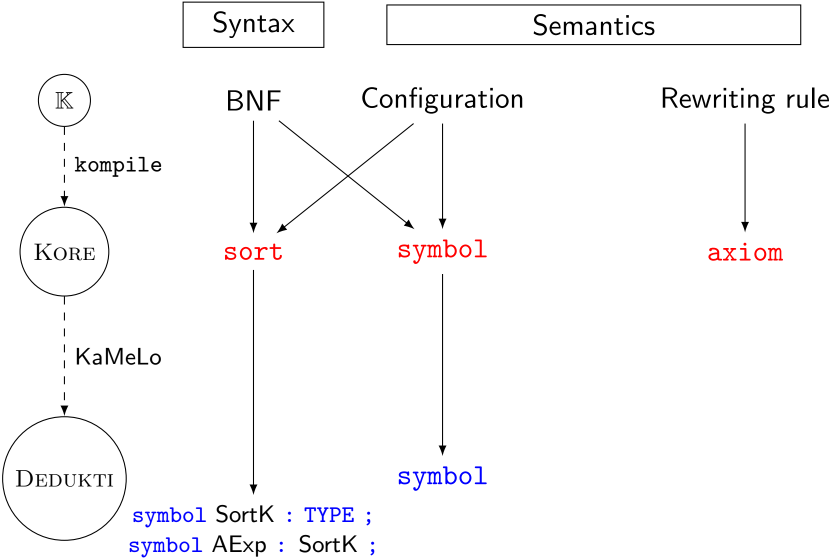
2. A deep encoding to check proofs in Dedukti
   - Translate Matching Logic constructors, notations and symbols
   - Translate Matching Logic proof system

3. Conclusion

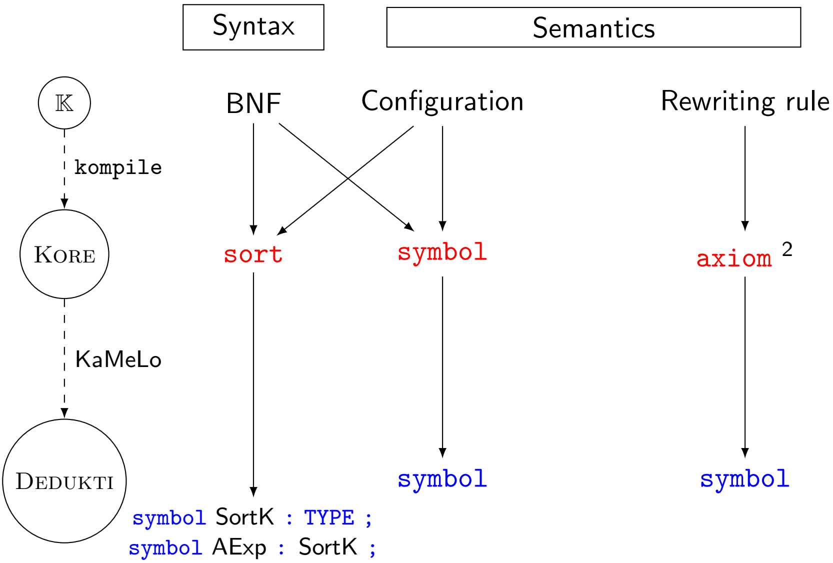
Translation from 𝕂 to DEDUKTI



Translation from 𝕂 to Dedukti



Translation from 𝕂 to Dedukti



[2] Matching Logic pattern

1. A shallow encoding to execute a program in Dedukti

2. A deep encoding to check proofs in Dedukti
   - Translate Matching Logic constructors, notations and symbols
   - Translate Matching Logic proof system

3. Conclusion

Matching Logic constructors

Matching Logic defines patterns

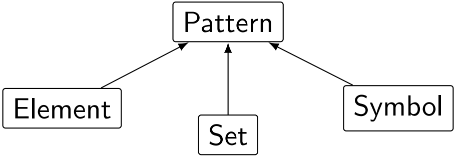
$\varphi ::= x \mid X \mid \sigma \mid \varphi \mathbin{@} \varphi \mid \bot \mid \varphi \to \varphi \mid \exists x.\varphi \mid \mu X.\varphi$

Matching Logic constructors

Matching Logic defines patterns

$\varphi ::= x \mid X \mid \sigma \mid \varphi \; @ \; \varphi \mid \bot \mid \varphi \to \varphi \mid \exists x.\varphi \mid \mu X.\varphi$

```
symbol #Pattern   : TYPE;
symbol #Element   : TYPE;
symbol #Set       : TYPE;
symbol #Symbol    : TYPE;
```



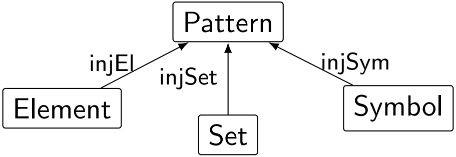
The next symbol:

```
symbol • : #Symbol; // Symbol
```

Matching Logic defines patterns

$\varphi ::= x \mid X \mid \sigma \mid \varphi \mathbin{@} \varphi \mid \bot \mid \varphi \rightarrow \varphi \mid \exists x.\varphi \mid \mu X.\varphi$

```
symbol #Pattern   : TYPE;
symbol #Element   : TYPE;
symbol #Set       : TYPE;
symbol #Symbol    : TYPE;
```



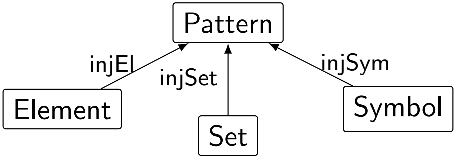

```
symbol injEl  : #Element → #Pattern;
symbol injSet : #Set → #Pattern;
symbol injSym : #Symbol → #Pattern;
```

Matching Logic constructors

Matching Logic defines patterns

$\varphi ::= x \mid X \mid \sigma \mid \varphi\ @\ \varphi \mid \bot \mid \varphi \to \varphi \mid \exists x.\varphi \mid \mu X.\varphi$

```
symbol #Pattern  : TYPE;
symbol #Element  : TYPE;
symbol #Set      : TYPE;
symbol #Symbol   : TYPE;
```




```
symbol injEl  : #Element → #Pattern;
symbol injSet : #Set → #Pattern;
symbol injSym : #Symbol → #Pattern;
```

```
symbol @_ML  : #Pattern → #Pattern → #Pattern;
symbol ⊥_ML  : #Pattern;
symbol ⇒_ML  : #Pattern → #Pattern → #Pattern;
symbol ∃_ML  : (#Element → #Pattern) → #Pattern;
symbol μ_ML  : (#Set → #Pattern) → #Pattern;
```

Notations vs Symbols

- Notations are syntactic sugar:

```
symbol ¬_ML : #Pattern → #Pattern;
rule ¬_ML $φ ↪ $φ ⇒_ML ⊥_ML;

symbol ∨_ML : #Pattern → #Pattern → #Pattern;
rule $φ0 ∨_ML $φ1 ↪ (¬_ML $φ0) ⇒_ML $φ1;
```

- Symbols are patterns:

```
symbol • : #Symbol; // Symbol
symbol ⇝ : #Pattern → #Pattern → #Pattern; // Notation
rule $φ1 ⇝ $φ2 ↪ $φ1 ⇒_ML ((injSym •) @_ML $φ2);
```

Matching Logic proof system

FOL Reasoning

$$\frac{}{\varphi \to (\psi \to \varphi)}\ \text{(Prop 1)}$$

$$\frac{}{(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))}\ \text{(Prop 2)}$$

$$\frac{}{((\varphi \to \bot) \to \bot) \to \varphi}\ \text{(Prop 3)}$$

$$\frac{\varphi_1 \quad \varphi_1 \to \varphi_2}{\varphi_2}\ \text{(Modus Ponens)}$$

$$\frac{}{\varphi[y/x] \to \exists x.\varphi}\ \text{(}\exists\text{-Quantifier)}$$

$$\frac{\varphi_1 \to \varphi_2 \quad (when\ x \notin FV(\varphi_2))}{(\exists x.\varphi_1) \to \varphi_2}\ \text{(}\exists\text{-Generalization)}$$

Technical rules

$$\frac{}{\exists x.x}\ \text{(Existence)}$$

$$\frac{}{\neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi])}\ \text{(Singleton)}$$

Frame Reasoning

$$\frac{}{C[\bot] \to \bot}\ (Propagation_\bot)$$

$$\frac{}{C[\varphi_1 \vee \varphi_2] \to C[\varphi_1] \vee C[\varphi_2]}\ (Propagation_\vee)$$

$$\frac{(when\ x \notin FV(C))}{C[\exists x.\varphi] \to \exists x.C[\varphi]}\ (Propagation_\exists)$$

$$\frac{\varphi_1 \to \varphi_2}{C[\varphi_1] \to C[\varphi_2]}\ \text{(Framing)}$$

Fixpoint Reasoning

$$\frac{\varphi}{\varphi[\psi/X]}\ \text{(Set Variable Substitution)}$$

$$\frac{}{\varphi[(\mu X.\varphi/X)] \to \mu X.\varphi}\ \text{(PreFixpoint)}$$

$$\frac{\varphi[\psi/X] \to \psi}{\mu X.\varphi \to \psi}\ \text{(Knaster-Tarski)}$$

Propositional fragment

$$\frac{}{\varphi \rightarrow (\psi \rightarrow \varphi)} \text{ (Prop 1)}$$

$$\frac{}{(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))} \text{ (Prop 2)}$$

$$\frac{}{((\varphi \rightarrow \bot) \rightarrow \bot) \rightarrow \varphi} \text{ (Prop 3)}$$

$$\frac{\varphi_1 \qquad \varphi_1 \rightarrow \varphi_2}{\varphi_2} \text{ (Modus Ponens)}$$

$\rightarrow$ No difficulty!

As usual:

```
injective symbol Prf : #Pattern → TYPE;
```

Propositional fragment

- $$\frac{}{\varphi \rightarrow (\psi \rightarrow \varphi)}\ (\text{Prop } 1)$$

```
symbol prop-1 : Π (φ ψ : #Pattern),
  Prf (φ ⇒_ML (ψ ⇒_ML φ));
```

Propositional fragment

- $$\frac{}{\varphi \to (\psi \to \varphi)} \text{ (Prop 1)}$$

```
symbol prop-1 : Π (φ ψ : #Pattern),
  Prf (φ ⇒_ML (ψ ⇒_ML φ));
```

- $$\frac{}{(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))} \text{ (Prop 2)}$$

  $\to$ **To be done as an exercise!**

- $$\frac{}{((\varphi \to \bot) \to \bot) \to \varphi} \text{ (Prop 3)}$$

  $\to$ **To be done as an exercise!**

- $$\frac{\varphi_1 \quad \varphi_1 \to \varphi_2}{\varphi_2} \text{ (Modus Ponens)}$$

```
symbol mp : Π (φ1 φ2 : #Pattern),
  Prf φ1 → Prf (φ1 ⇒_ML φ2) → Prf φ2;
```

A Matching Logic proof encoded into Dedukti

$$
\dfrac{\dfrac{}{\Gamma \vdash \varphi \to \alpha}\,(\text{P1}) \qquad \dfrac{\dfrac{}{\Gamma \vdash \varphi \to (\alpha \to \varphi)}\,(\text{P1}) \qquad \dfrac{}{\Gamma \vdash (\varphi \to (\alpha \to \varphi)) \to ((\varphi \to \alpha) \to \alpha)}\,(\text{P2})}{\Gamma \vdash (\varphi \to \alpha) \to \alpha}\,(\text{MP})}{\Gamma \vdash \alpha}\,(\text{MP})
$$

avec $\alpha \equiv \varphi \to \varphi$

```
symbol imp-identity : Π φ0, Prf (φ0 ⇒ML φ0) :=
  λ φ0,
    mp (φ0 ⇒ML (φ0 ⇒ML φ0))
       (φ0 ⇒ML φ0)
       (prop-1 φ0 φ0)
       (mp (φ0 ⇒ML ((φ0 ⇒ML φ0) ⇒ML φ0))
           ((φ0 ⇒ML (φ0 ⇒ML φ0)) ⇒ML (φ0 ⇒ML φ0))
           (prop-1 φ0 (φ0 ⇒ML φ0))
           (prop-2 φ0 (φ0 ⇒ML φ0) φ0));
```

FOL reasoning

$$\frac{}{\varphi[y/x] \quad \rightarrow \quad \exists x.\varphi} \ (\exists\text{-Quantifier})$$

$$\frac{\varphi_1 \quad \rightarrow \quad \varphi_2 \qquad (\textit{when } x \notin FV(\varphi_2))}{(\exists x.\varphi_1) \quad \rightarrow \quad \varphi_2} \ (\exists\text{-Generalization})$$

FOL reasoning

$$\frac{}{\varphi[y/x] \quad \rightarrow \quad \exists x.\varphi} \ (\exists\text{-Quantifier})$$

$$\frac{\varphi_1 \quad \rightarrow \quad \varphi_2 \qquad (\textit{when } x \notin FV(\varphi_2))}{(\exists x.\varphi_1) \quad \rightarrow \quad \varphi_2} \ (\exists\text{-Generalization})$$

Problems:

- Substitution
- Checking of free variable

FOL reasoning

$$\frac{}{\varphi[y/x] \quad \rightarrow \quad \exists x.\varphi} \; (\exists\text{-Quantifier})$$

$$\frac{\varphi_1 \quad \rightarrow \quad \varphi_2 \qquad (\textit{when } x \notin FV(\varphi_2))}{(\exists x.\varphi_1) \quad \rightarrow \quad \varphi_2} \; (\exists\text{-Generalization})$$

Problems:

- Substitution
- Checking of free variable

→ Solution: HOAS

FOL reasoning

- $$\frac{}{\varphi[y/x] \quad \rightarrow \quad \exists x.\varphi} \text{ (}\exists\text{-Quantifier)}$$

```
symbol ex-quantifier :
  Π(φ : #Element → #Pattern)
   (y : #Element),
  Prf (φ y ⇒_ML (∃_ML φ)) ;
```

Very close to $\frac{}{\varphi \quad \rightarrow \quad \exists x.\varphi}$ ($\exists$-Quantifier)
because α-renaming is done by the DEDUKTI binder.

FOL reasoning

- $$\frac{}{\varphi[y/x] \ \rightarrow \ \exists x.\varphi} \ (\exists\text{-Quantifier})$$

```
symbol ex-quantifier :
  Π(φ : #Element → #Pattern)
   (y : #Element),
  Prf (φ y ⇒_ML (∃_ML φ)) ;
```

- $$\frac{\varphi_1 \ \rightarrow \ \varphi_2 \quad (\textit{when } x \notin FV(\varphi_2))}{(\exists x.\varphi_1) \ \rightarrow \ \varphi_2} \ (\exists\text{-Generalization})$$

```
symbol ex-generalization :
  Π (φ1 : #Element → #Pattern)
    (φ2 : #Pattern),
  (Π (x : #Element), Prf (φ1 x ⇒_ML φ2) )
       → Prf ( (∃_ML φ1) ⇒_ML φ2 ) ;
```

Framing reasoning

$$\frac{}{C[\bot] \;\rightarrow\; \bot}\;(Propagation_{\bot})$$

$$\frac{}{C[\varphi_1 \;\vee\; \varphi_2] \;\rightarrow\; C[\varphi_1] \;\vee\; C[\varphi_2]}\;(Propagation_{\vee})$$

$$\frac{\varphi_1 \;\rightarrow\; \varphi_2}{C[\varphi_1] \;\rightarrow\; C[\varphi_2]}\;(\text{Framing})$$

$$\frac{(when\ x \notin FV(C))}{C[\exists x.\varphi] \;\rightarrow\; \exists x.C[\varphi]}\;(Propagation_{\exists})$$

Framing reasoning

$$\frac{}{C[\bot] \;\rightarrow\; \bot}\ (\mathit{Propagation}_\bot)$$

$$\frac{}{C[\varphi_1 \;\vee\; \varphi_2] \;\rightarrow\; C[\varphi_1] \;\vee\; C[\varphi_2]}\ (\mathit{Propagation}_\vee)$$

$$\frac{\varphi_1 \;\rightarrow\; \varphi_2}{C[\varphi_1] \;\rightarrow\; C[\varphi_2]}\ (\text{Framing})$$

$$\frac{(\mathit{when}\ x \notin FV(C))}{C[\exists x.\varphi] \;\rightarrow\; \exists x.C[\varphi]}\ (\mathit{Propagation}_\exists)$$

Problem:

- Application context $C ::= \square \mid C \ @\ \varphi \mid \varphi\ @\ C$

Frame reasoning - Application context

Model the BNF grammar $C ::= \square \mid C \ @ \ \varphi \mid \varphi \ @ \ C$:

```
symbol #AC : TYPE ;
symbol HOLE : #AC ;
symbol ACleft  : #AC → #Pattern → #AC ;
symbol ACright : #Pattern → #AC → #AC ;
```

Frame reasoning - Application context

Model the BNF grammar $C ::= \Box \mid C \mathbin{@} \varphi \mid \varphi \mathbin{@} C$:

```
symbol #AC : TYPE ;
symbol HOLE : #AC ;
symbol ACleft  : #AC → #Pattern → #AC ;
symbol ACright : #Pattern → #AC → #AC ;
```

$$\frac{}{C[\bot] \;\rightarrow\; \bot}\ (\mathit{Propagation}_\bot)$$

Frame reasoning - Application context

Model the BNF grammar $C ::= \square \mid C \ @\ \varphi \mid \varphi\ @\ C$:

```
symbol #AC : TYPE ;
symbol HOLE : #AC ;
symbol ACleft  : #AC → #Pattern → #AC ;
symbol ACright : #Pattern → #AC → #AC ;
```

Translate an application context into a pattern:

```
symbol AC2P : #AC → #Pattern → #Pattern ;
rule AC2P HOLE $x ↪ $x ;
rule AC2P (ACleft  $C $P) $x ↪ (AC2P $C $x) @ML $P ;
rule AC2P (ACright $P $C) $x ↪ $P @ML (AC2P $C $x) ;
```

Frame reasoning - Application context

Model the BNF grammar $C ::= \square \mid C \mathbin{@} \varphi \mid \varphi \mathbin{@} C$:

```
symbol #AC : TYPE ;
symbol HOLE : #AC ;
symbol ACleft  : #AC → #Pattern → #AC ;
symbol ACright : #Pattern → #AC → #AC ;
```

Translate an application context into a pattern:

```
symbol AC2P : #AC → #Pattern → #Pattern ;
rule AC2P HOLE $x ↪ $x ;
rule AC2P (ACleft  $C $P) $x ↪ (AC2P $C $x) @ML $P ;
rule AC2P (ACright $P $C) $x ↪ $P @ML (AC2P $C $x) ;
```

Translate the rule $\dfrac{}{C[\bot] \ \rightarrow \ \bot}\ (Propagation_{\bot})$:

```
symbol propag-bot :
  Π(C : #AC), Prf (AC2P C ⊥ML ⇒ML ⊥ML);
```

```
type propag-bot (ACright (injSym •) HOLE) ;
  // Prf ((injSym •) @ML ⊥ML ⇒ML ⊥ML)
```

Frame reasoning - Rules

- $\dfrac{}{C[\bot] \;\rightarrow\; \bot}$ $(Propagation_{\bot})$ → **Already done!**
- $\dfrac{}{C[\varphi_1 \;\vee\; \varphi_2] \;\rightarrow\; C[\varphi_1] \;\vee\; C[\varphi_2]}$ $(Propagation_{\vee})$

  → **To be done as an exercise!**
- $\dfrac{\varphi_1 \;\rightarrow\; \varphi_2}{C[\varphi_1] \;\rightarrow\; C[\varphi_2]}$ (Framing) → **To be done as an exercise!**
- $\dfrac{(when\ x \notin FV(C))}{C[\exists x.\varphi] \;\rightarrow\; \exists x.C[\varphi]}$ $(Propagation_{\exists})$

  → **Combine HOAS + Application context**

Fixpoint reasoning

$$\frac{\varphi}{\varphi[\psi/X]} \text{ (Set Variable Substitution)}$$

$$\frac{}{\varphi[(\mu X.\varphi)/X] \;\rightarrow\; \mu X.\varphi} \text{ (PreFixpoint)}$$

$$\frac{\varphi[\psi/X] \;\rightarrow\; \psi}{(\mu X.\varphi) \;\rightarrow\; \psi} \text{ (Knaster-Tarski)}$$

Problem:

- Is there a problem?

Fixpoint reasoning

- $$\frac{\varphi}{\varphi[\psi/X]}$$ (Set Variable Substitution)
- $$\frac{}{\varphi[(\mu X.\varphi)/X] \;\rightarrow\; \mu X.\varphi}$$ (PreFixpoint)

```
symbol Pre-fixpoint :
  Π (φ : #Pattern → #Pattern),
  Prf ( φ (μ_ML φ) ⇒_ML (μ_ML φ) ) ;
```

- $$\frac{\varphi[\psi/X] \;\rightarrow\; \psi}{(\mu X.\varphi) \;\rightarrow\; \psi}$$ (Knaster-Tarski)

```
symbol Knaster-Tarski :
  Π(φ : #Pattern → #Pattern)
   (ψ : #Pattern),
  Prf ( φ ψ ⇒_ML ψ ) →
  Prf ( (μ_ML φ) ⇒_ML ψ ) ;
```

```
symbol μ_ML : (#Pattern → #Pattern) → #Pattern;
```

Fixpoint reasoning

- $\dfrac{\varphi}{\varphi[\psi/X]}$ (Set Variable Substitution) $\rightarrow$ **X is a free variable!**
- $\dfrac{}{\varphi[(\mu X.\varphi)/X] \;\rightarrow\; \mu X.\varphi}$ (PreFixpoint)

```
symbol Pre-fixpoint :
  Π (φ : #Pattern → #Pattern),
  Prf ( φ (μ_ML φ) ⇒_ML (μ_ML φ) ) ;
```

- $\dfrac{\varphi[\psi/X] \;\rightarrow\; \psi}{(\mu X.\varphi) \;\rightarrow\; \psi}$ (Knaster-Tarski)

```
symbol Knaster-Tarski :
  Π(φ : #Pattern → #Pattern)
   (ψ : #Pattern),
  Prf ( φ ψ ⇒_ML ψ ) →
  Prf ( (μ_ML φ) ⇒_ML ψ ) ;
```

```
symbol μ_ML : (#Pattern → #Pattern) → #Pattern;
```

The last problem

- $\dfrac{\varphi}{\varphi[\psi/X]}$ (Set Variable Substitution)

```
symbol Set-var-subst :
  Π (φ ψ : #Pattern) (n : nat),
    Prf φ → Prf (subst φ ψ n) ;
```

where the free variable is modelled by:

```
symbol Free : nat → #Set;
```

and the substitution is modelled by:

```
symbol subst :
  #Pattern → #Pattern → nat → #Pattern; // φ[ψ/X]

rule subst (injEl $x) _ _ ↪ injEl $x;

rule subst (injSet (Free $m)) $ψ $n ↪
 ite (eq $m $n) $ψ (injSet (Free $m));
```

Technical rules

$$\frac{}{\exists x.x}\ \text{(Existence)}$$

$$\frac{}{\neg(C_1[x \ \wedge \ \varphi] \ \wedge \ C_2[x \ \wedge \ \neg\varphi])}\ \text{(Singleton)}$$

To be done as an exercise!

To remember

Computational part of an embedding

- Use rewriting rules!
  - Be careful about the expressivity of rewriting system!
  - Be careful to keep the confluence!
  - Be careful to keep the termination!

Deductive part of an embedding

- Model the provability relation: `symbol` Prf : #Pattern → `TYPE`
- Model variables and binders: HOAS vs De Bruijn indices
- Model a grammar:
  - type as set
  - symbol as constructor
- Model a deduction rule: symbol