How to write a translator to Dedukti
The case of Agda

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How to write a translator to Dedukti

Previous talks. How to define theories and write proofs in Dedukti (eg. the theory $\mathcal{U}$).

This talk. How to write an automatic translator from a proof assistant to Dedukti:

- General principles on writing such a translator
- Specific case of the Agda2Dedukti translator
1. Principles on translating from a proof assistant to Dedukti

2. What is Agda?

3. Encoding Agda in Dedukti

4. Implementation of Agda2Dedukti

5. Inductive types and dependent pattern matching

6. Universe polymorphism

7. Eta equality & irrelevance

8. Conclusion
How to translate from a proof assistant to Dedukti

Step 0. Find/define a system $\mathcal{O}$ corresponding to the proof assistant’s logic (not easy!)

Step 1. Define a Dedukti theory $D[\mathcal{O}] = (\Sigma, \mathcal{R})$ representing the object logic in Dedukti.

Step 2. Define a translation $\llbracket - \rrbracket : \Lambda_{\mathcal{O}} \rightarrow \Lambda_{DK}$. The pair $(D[\mathcal{O}], \llbracket - \rrbracket)$ is an encoding of $\mathcal{O}$.

Step 3. Implement the translating function, making use of the APIs and other tools offered by the proof assistant.
Not all encodings are created equal

- An encoding is **sound** if:
  \[ \vdash_{\mathcal{O}} M : A \quad \text{implies} \quad \vdash_{D[\mathcal{O}]} \llbracket M \rrbracket : EI \llbracket A \rrbracket \]

- An encoding is **conservative** if:
  \[ \vdash_{D[\mathcal{O}]} M : EI \llbracket A \rrbracket \quad \text{implies} \quad \exists N, \quad \vdash_{\mathcal{O}} N : A \]

- An encoding is **adequate** if for each type \( A \):
  \( \llbracket - \rrbracket \) is a *compositional bijection* between \( A \) and \( EI \llbracket A \rrbracket \)
Nor are all proof assistants equal

The difficulty of encoding (the core language of) a proof assistant depends on its features:

**Dependent types** are in Coq, Agda, Lean, ...

**Inductive types** are in most proof assistants.\(^1\)

**Universe polymorphism** is in Coq, Agda, Lean, ...

**Impredicativity** is in all proof assistants, except Agda and Epigram.

**Eta-equality & irrelevance** are present in different shapes in different proof assistants.

---

\(^1\)Most type-theoretic proof assistants also support inductive families.
Neither are their implementations

The difficulty of writing a translator also depends on the *implementation* of the proof assistant:

- In systems based on Curry-Howard (Coq/Agda/Matita), proof terms are already *in the internal syntax*, so are easier to translate.
- In LCF-like assistants (Isabelle/HOL), there are no proof terms, so we need to *reconstruct* them from proof derivations.
- In other systems (PVS), proofs derivations are not even internally available.\(^2\)

\(^2\)See Gabriel’s talk for a solution.
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What is Agda?

Agda is a dependently typed programming language and proof assistant based on Martin-Löf type theory.

It has indexed datatypes, dependent pattern matching, and explicit universe polymorphism.

Its type checker identifies terms up to $\beta$-equality and $\eta$-equality for functions and records, and supports definitional proof irrelevance.
Data types in Agda

```haskell
data _⊎_ (A B : Set) : Set where
  left  : A → A ⊎ B
  right : B → A ⊎ B

data _≤_ : ℕ → ℕ → Set where
  ≤-zero : ∀ {n} → zero ≤ n
  ≤-suc : ∀ {m n} → m ≤ n → suc m ≤ suc n
```
Pattern matching in Agda

\[ \_<_\_ : \mathbb{N} \to \mathbb{N} \to \text{Set} \]
\[ m < n = m \leq \text{suc } n \]

\begin{align*}
\text{compare} : (m \ n : \mathbb{N}) & \to (m \leq n) \uplus (n < m) \\
\text{compare} \ \text{zero} \ n & = \text{left } \leq\text{-zero} \\
\text{compare} (\text{suc } m) \ \text{zero} & = \text{right } \leq\text{-zero} \\
\text{compare} (\text{suc } m) (\text{suc } n) \ & \text{ with compare } m \ n \\
... \ & | \ \text{left } m \leq n \ = \ \text{left } (\leq\text{-suc } m \leq n) \\
... \ & | \ \text{right } n < m \ = \ \text{right } (\leq\text{-suc } n < m)
\end{align*}
Agda as a PTS

At its core, Agda is a pure type system with sorts $\text{Set } \ell$ where $\ell$ is a universe level.

\[
U : (\ell : \text{Level}) \rightarrow \text{Set } (\text{lsuc } \ell) \\
U \ell = \text{Set } \ell
\]

\[
\text{prod} : (\ell_1 \ell_2 : \text{Level}) \\
( A : \text{Set } \ell_1) \ ( B : A \rightarrow \text{Set } \ell_2) \\
\rightarrow \text{Set } (\ell_1 \sqcup \ell_2) \\
\text{prod } _-_ _ A B = (x : A) \rightarrow B \ x
\]
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Encoding Agda terms in Dedukti

Variable  \[ [x] = \]
Def. symbol  \[ [f] = \]
Constructor  \[ [D.c] = \]
Lambda  \[ [\lambda x \to u] = \]
Application  \[ [u \ v] = \]
Pi type  \[ [(x : A) \to B] = \]
Universe  \[ [\text{Set } \ell] = \]
Encoding Agda terms in Dedukti

Variable

\[ [x] = x \]

Def. symbol

\[ [f] = f \]

Constructor

\[ [D.c] = D\_\_c \]

Lambda

\[ [\lambda x \to u] = \]

Application

\[ [u \; v] = \]

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Universe

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### Encoding Agda terms in Dedukti

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<tr>
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Agda uses **Russell-style universes**: Elements are *types* themselves.

\[
A : \text{Set}_I \\
\overline{\quad}
\quad A \quad \text{TYPE}
\]

In Dedukti, if \( A : \text{Set} \), we cannot have \( a : A \). Thus, Dedukti uses a form of **Tarski-style universes**: Elements are *codes* that can be *interpreted* as types.

\[
c : \text{U} \left( \text{set } I \right) \\
\overline{\quad}
\quad \text{El} \left( \text{set } I \right) \quad c \quad \text{TYPE}
\]

---

\(^3\)https://www.cs.rhul.ac.uk/home/zhao/universes.pdf
Encoding Agda’s PTS in Dedukti

Sort : Type.
set : Lvl -> Sort.

U : (s : Sort) -> Type.
def El : (s : Sort) -> (a : U s) -> Type.

def axiom : Sort -> Sort.
[i] axiom (set i) --> set (s i).

def rule : Sort -> Sort -> Sort.
[i, j] rule (set i) (set j) --> set (max i j).

(We postpone the definition of Lvl until later,
for now you can assume lvl = \mathbb{N}.)

Encoding pi types

- Add a constant $\text{prod}$ for encoding the pi type:

$$
\begin{align*}
A : U \ s_A & \quad B : \text{El} \ s_A \ A \rightarrow U \ s_B \\
\text{prod} \ s_A \ s_B \ A \ B & : U \ (\text{rule} \ s_A \ s_B)
\end{align*}
$$
Encoding pi types

- Add a constant \( \text{prod} \) for encoding the pi type:

\[
A : \text{U} \ s_A \quad B : \text{El} \ s_A \ A \rightarrow \text{U} \ s_B \\
\text{prod} \ s_A \ s_B \ A \ B : \text{U} \ (\text{rule} \ s_A \ s_B)
\]

- Identify elements of \( \text{prod} \) with the *metatheoretic arrow type*:

\[
\text{El } _\_ \ (\text{prod} \ s_A \ s_B \ A \ B) \\
= (x : \text{El} \ s_A \ A) \rightarrow \text{El} \ s_B \ (B \ x)
\]
Encoding pi types in Dedukti

\[ \text{prod} : (s_A : \text{Sort}) \rightarrow (s_B : \text{Sort}) \rightarrow (A : U s_A) \rightarrow (B : (\text{El} s_A A \rightarrow U s_B)) \rightarrow U (\text{rule} s_A s_B). \]

\[ [s_A, s_B, A, B] \]
\[ \text{El} \_ (\text{prod} s_A s_B A B) \]
\[ \rightarrow (x : \text{El} s_A A) \rightarrow \text{El} s_B (B x). \]
Reconstructing sorts

For translating pi types, we need access to the sort of the domain and codomain.

Luckily, Agda’s type checker already annotates each type $A$ with its sort $s(A)$.

**Examples.** $s(\mathbb{N}) = \text{Set}$, $s(\text{Set}) = \text{Set}_1$, $s(\text{Set}_1 \to \text{Set}) = \text{Set}_2$
# Encoding Agda terms in Dedukti

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Encoding Agda terms in Dedukti

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Constructor \[ [D.c] = D_{\_\_c} \]

Lambda \[ [\lambda x \to u] = x \Rightarrow [u] \]

Application \[ [u \; v] = [u] \; [v] \]

Pi type \[ [\,(x : A) \to B] = \text{prod} \; |s(A)| \; |s(B)| \]
\[ [A] \; (x \Rightarrow [B]) \]

where \[ |\text{Set} \; \ell| = \text{set} \; [\ell] \]

Universe \[ [\text{Set} \; \ell] = ??? \]

(We will see how to translate levels later.)
Encoding universe codes

- Add a constant $u$ for encoding the $\textbf{Set}$ type:

$$
\begin{align*}
  s &: \text{Sort} \\
  u \ s &: U \ (\text{axiom } s)
\end{align*}
$$
Encoding universe codes

- Add a constant $u$ for encoding the $\text{Set}$ type:

\[
\begin{align*}
    s &: \text{Sort} \\
    u \cdot s &: \aleph (\text{axiom } s)
\end{align*}
\]

- Identify elements of $u \cdot s$ with the ones of $\aleph s$:

\[
\text{El } _\_ (u \cdot s) = \aleph s
\]

In Dedukti:

\[
u : (s : \text{Sort}) \rightarrow \aleph (\text{axiom } s).
\]

\[
[i] \text{El } _\_ (u \cdot s) \rightarrow \aleph s.
\]
Encoding Agda terms in Dedukti

Variable

\[ [x] = x \]

Def. symbol

\[ [f] = f \]

Constructor

\[ [D.c] = D__c \]

Lambda

\[ [\lambda x \rightarrow u] = x \Rightarrow [u] \]

Application

\[ [u \; v] = [u] \; [v] \]

Pi type

\[ [(x : A) \rightarrow B] = \text{prod} \; |s(A)| \; |s(B)| \]
\[ [A] \; (x \Rightarrow [B]) \]

where \[ |\text{Set} \; \ell| = \text{set} \; [\ell] \]

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(We will see how to translate levels later.)
Encoding Agda terms in Dedukti

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\([x] = x\)

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\([D.c] = D_{\_\_c}\)

Lambda
\([\lambda x \rightarrow u] = x \Rightarrow [u]\)

Application
\([u \; v] = [u] \; [v]\)

Pi type
\([((x : A) \rightarrow B)] = \text{prod} |s(A)| \; |s(B)|
\quad [A] \; (x \Rightarrow [B])\)

where \(|\text{Set} \; \ell| = \text{set} \; [\ell]\

Universe
\([\text{Set} \; \ell] = u \; (\text{set} \; [\ell])\)

(We will see how to translate levels later.)
Encoding Agda definitions in Dedukti

Data types (no parameters or indices)

\[
\begin{array}{c}
\text{data } D : U \text{ where } \\
\quad c : A
\end{array}
\] =\[
\begin{array}{c}
D : \text{El } | s(U) | [U] . \\
D__c : \text{El } | U | [A] .
\end{array}
\]

Function definitions (no pattern matching)

\[
\begin{array}{c}
f : A \\
f \ x = v
\end{array}
\] = \[
\begin{array}{c}
def \ f : \text{El } | s(A) | [A] . \\
\ 
\[x\] \ f \ x \rightarrow [v] .
\end{array}
\]
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Implementation of Agda2Dedukti

Agda2Dedukti is implemented as an Agda backend. This allows us to reuse parts of Agda’s implementation:

- Internal syntax representation
- Type checking monad TCM
Structure of the Agda typechecker

lexer & parser

Concrete syntax

scope checker

Abstract syntax

type checker

Internal syntax

optimizer

Treeless syntax

MAlonzo

.hs file

Binary
Structure of the Agda typechecker

lexer & parser

Concrete syntax

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type checker

Internal syntax

Agda2Dk

optimizer

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MAlonzo

.Treeless syntax

GHC

.HS file

Binary
Agda’s internal syntax

data Term
  = Var Int Elims
    | Lam ArgInfo (Abs Term)
    | Lit Literal
    | Def QName Elims
    | Con ConHead ConInfo Elims
    | Pi (Dom Type) (Abs Type)
    | Sort Sort
    | Level Level
    | MetaV MetaId Elims
    | DontCare Term
    | Dummy String Elims

--- x u v ..
--- \( \lambda x \rightarrow v \)
--- 42, 'a', ...
--- f u v ..
--- c u v ..
--- \((x : A) \rightarrow B\)
--- \textit{Set, Set}_1, \textit{Prop}, ...
--- lzero, ...
--- _X_235

\(^4\)Code from \texttt{Agda.Syntax.Internal}
Agda’s TCM monad

Agda’s typechecker uses a type-checking monad TCM:

type TCM a
getConstInfo :: QName -> TCM Definition
getBuiltin :: String -> TCM Term
getContext :: TCM Context
addContext :: (Name, Dom Type) -> TCM a -> TCM a
checkInternal :: Term -> Type -> TCM ()
reconstructParameters :: Type -> Term -> TCM Term
...

Putting it all together

example : \((1 \leq 2) \uplus (2 < 1)\)
example = left \((\leq\text{-suc} \leq\text{-zero})\)

\{\|!\_\uplus\_\_left\|\}
\(\{\|!\_\leq\_\|\}\)
\((\text{Nat\_suc Nat\_zero})\)
\((\text{Nat\_suc (Nat\_suc Nat\_zero)})\))
\(\{\|!\_<\_\|\}\)
\((\text{Nat\_suc (Nat\_suc Nat\_zero)})\)
\((\text{Nat\_suc Nat\_zero})\))
\(\{\|!\_\leq\_\_\_\_\_\leq\_\text{-suc}\|\}\)
\text{Nat\_zero}
\((\text{Nat\_suc Nat\_zero})\)
\((\text{Nat\_suc Nat\_zero})\))
\(\{\|!\_\leq\_\_\_\_\_\leq\_\text{-zero}\|\}\) \((\text{Nat\_suc Nat\_zero})\)))
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Translating datatypes and constructors to constants

Data types and their constructors do not reduce, so we translate them to constants in Dedukti.

Example. \(\leq\) is translated to:

\[
\{\|\_\_\leq\_\_\}\ : \ El (set (s 0)) (prod (set 0) (set (s 0)) \\
\quad \Nat (_0 \Rightarrow (prod (set 0) (set (s 0)) \\
\quad \quad \Nat (_0 \Rightarrow (u (set 0))))))).
\]

\[
\{\|\_\_\leq\_\_\_-\text{zero}\}\ : \ El (set 0) (prod (set 0) (set 0) \\
\quad \Nat (n \Rightarrow (\{\|\_\_\leq\_\_\}\ \Nat\_\_\text{zero} \ n))).
\]

\[
\{\|\_\_\leq\_\_\_-\text{suc}\}\ : \ El (set 0) (prod (set 0) (set 0) \Nat \\
\quad (m \Rightarrow (prod (set 0) (set 0) \\
\quad \Nat (n \Rightarrow (prod (set 0) (set 0) \\
\quad \quad (\{\|\_\_\leq\_\_\}\ m n) \\
\quad \quad (_0 \Rightarrow (\{\|\_\_\leq\_\_\}\ (\Nat\_\_\text{suc} m) (\Nat\_\_\text{suc} n))))))))).
\]
Reconstruction of data parameters

Constructors in Agda do \textit{not} store their parameters.

Reconstructing parameters requires a \textit{type-directed traversal} of the syntax.

We can reuse Agda’s \texttt{reconstructParameters}, which does exactly this!
Filling implicit arguments &
reconstructing parameters

left (≤-suc ≤-zero) : (1 ≤ 2) ⊔ (2 < 1)
Filling implicit arguments & reconstructing parameters

Agda’s type checker infers implicit arguments during type checking.

\[
\text{left } (\leq\text{-suc } \leq\text{-zero}) : (1 \leq 2) \UPlus (2 < 1) \\
\downarrow \\
\text{left } (\leq\text{-suc } \{m = 0\} \{n = 1\} (\leq\text{-zero } \{n = 1\}))
\]
Filling implicit arguments & reconstructing parameters

Agda’s type checker infers implicit arguments during type checking.

Agda2Dk makes all implicit arguments explicit and reconstructs constructor parameters.

\[
\begin{align*}
\text{left } (\leq \text{-suc} \leq \text{-zero}) & : (1 \leq 2) \uplus (2 < 1) \\
\Downarrow & \\
\text{left } (\leq \text{-suc} \{m = 0\} \{n = 1\} (\leq \text{-zero} \{n = 1\})) & \Downarrow \\
\text{left } (1 \leq 2) (2 < 1) (\leq \text{-suc} 0 1 (\leq \text{-zero} 1))
\end{align*}
\]
Translating clauses to rewrite rules

Functions in Agda are defined by a set of clauses, so we translate them to a constant + a set of rewrite rules.

Example. \texttt{compare} is translated to:

\begin{verbatim}
def compare : El (set 0) (prod (set 0) (set 0))
  Nat (m => (prod (set 0) (set 0))
    Nat (n => (||_∪_|||_|} (||_≤_|||_|} m n) (||<_|||_|} n m)))))).

[n] compare \texttt{Nat__zero} n -->
  {||∪_|||_|} (||≤_|||_|} \texttt{Nat__zero} n)
  (||<_|||_|} n \texttt{Nat__zero} (||≤_|||_|} ≤-zero|} n).

[m] compare (\texttt{Nat__suc} m) \texttt{Nat__zero} -->
  {||∪_|||_|} (||≤_|||_|} (\texttt{Nat__suc} m) \texttt{Nat__zero})
  (||<_|||_|} \texttt{Nat__zero} (\texttt{Nat__suc} m))
  (||≤_|||_|} ≤-zero|} (\texttt{Nat__suc} (\texttt{Nat__suc} m))).

[m, n] compare (\texttt{Nat__suc} m) (\texttt{Nat__suc} n) -->
  {||with-66|} m n (compare m n).
\end{verbatim}
Drawbacks of generating rewrite rules

Generating a new rewrite rule for each clause means that we are extending the theory with each definition.

Moreover, checking correctness (completeness & termination) of rewrite rules is very hard.

**Ongoing work:** Instead, we can translate definitions by pattern matching to eliminators.\(^5\)

```
def compare := Nat__ind...
```

---

\(^5\)Ask Thiago for details!
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Universe polymorphism

Sometimes one wishes to use a definition at multiple universes (e.g. List Nat but also List Set_0).
Universe polymorphism

Sometimes one wishes to use a definition at multiple universes (e.g. List Nat but also List Set₀).

**Bad solution.** Define a new Listᵢ for each level i.
Universe polymorphism

Sometimes one wishes to use a definition at multiple universes (e.g. List Nat but also List Set_0).

**Bad solution.** Define a new List_i for each level i.

**Universe polymorphism** allows definitions that can be used at multiple universe levels:

```haskell
data List {i} (A : Set i) : Set i where
  [] : List A
  _::_ : A → List A → List A

map : {i j : Level} → {A : Set i} → {B : Set j} → (f : A → B) → List A → List B
map f [] = []
map f (x :: l) = f x :: map f l
```
Other forms of universe polymorphism

Universe polymorphism in Agda is very different from universe polymorphism in Coq:

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<td>No</td>
</tr>
<tr>
<td>Cumulativity</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>((\text{Set}<em>i \subseteq \text{Set}</em>{i+1}))</td>
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In this talk we only see the encoding of Agda’s universe polymorphism.

For Coq’s version, see Gaspard Ferey’s PhD thesis.
Idea. Generalize the encoding of the arrow type:

setOmega : Sort.

forall : (l : (Lvl -> Sort)) ->
((i : Lvl) -> U (l i)) -> U setOmega.

[l, t] El _ (forall l t) -->
(i : Lvl) -> El (l i) (t i).
Universe polymorphism in Dedukti

Idea. Generalize the encoding of the arrow type:

\[
\text{setOmega} : \text{Sort}.
\]

\[
\text{forall} : (l : (\text{Lvl} \to \text{Sort})) \to (((i : \text{Lvl}) \to \text{U}(l \; i)) \to \text{U} \text{setOmega}.
\]

\[
[l, t] \; \text{El}_\_ \; (\text{forall} \; l \; t) \to
(i : \text{Lvl}) \to \text{El} \; (l \; i) \; (t \; i).
\]

We extend the translation function with:

Level quantification

\[
\left[(i : \text{Level}) \to A\right] = \forall \,(i \to [s(A)])
\]

Level application

\[
[M \; l] = [M] \; [l]
\]

Level abstraction

\[
[\lambda i. M] = i \Rightarrow [M]
\]
Now the constant `List` can be given the type:

\[
\text{El set}\Omega \\
(\forall i \rightarrow \text{set}\ (\text{suc } i)) \\
(\forall i \rightarrow \text{prod}\ (\text{set}\ (\text{suc } i)) \\
(\text{set}\ (\text{suc } i)) \\
(\text{u}\ (\text{set } i)) \\
(\_ \rightarrow \text{u}\ (\text{set } i)))
\]

Which, as expected, computes to:

\[
(i : \text{Lvl}) \rightarrow \text{U}\ (\text{set } i) \rightarrow \text{U}\ (\text{set } i)
\]
Universe levels

Levels are given by the syntax:

\[ l, l_1, l_2 ::= i \mid l_{\text{zero}} \mid l_{\text{suc}} \mid l_1 \sqcup l_2. \]
Universe levels

Levels are given by the syntax:

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Universe levels

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Distributivity: \( l\text{succ } (a \sqcup b) = l\text{succ } a \sqcup l\text{succ } b \)

Neutrality: \( a \sqcup l\text{zero} = a \)

Subsumption: \( a \sqcup l\text{succ}^n a = l\text{succ}^n a \)
The challenge of representing universe polymorphism

To establish the encoding’s soundness,

\[ l_1 \equiv l_2 \text{ should imply } \llbracket l_1 \rrbracket \equiv \llbracket l_2 \rrbracket \]

Possible solutions:
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   *Works well, but there is a catch (next slide).*

3. Decision procedure integrated in Dedukti?
   *We leave this to the future generations.*
Current solution: levels as sets

Idea. Every level $l$ admits a unique canonical form

$$l = \max\{n, i_1 + m_1, \ldots, i_k + m_k\}$$

where $i_1, \ldots, i_k \in FV(l)$, $n, m_1, \ldots, m_k \in \mathbb{N}$ and $m_j \leq n$. 
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$$\max\{i + n, i + m\} = i + \max\{n, m\}$$
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But idempotence and subsumption require a non-linear rule:

$$\max \{ i + n, i + m \} = i + \max \{ n, m \}$$

This breaks confluence of pre-terms, and prevents proving conservativity.
From Agda to Dedukti

1. Principles on translating from a proof assistant to Dedukti
2. What is Agda?
3. Encoding Agda in Dedukti
4. Implementation of Agda2Dedukti
5. Inductive types and dependent pattern matching
6. Universe polymorphism
7. Eta equality & irrelevance
8. Conclusion
Eta equality in Agda

Agda supports two kinds of eta-equality:

1. Eta for functions:

   \[
   f : (x : A) \rightarrow B \\
   f = (\lambda x \rightarrow f \, x) : (x : A) \rightarrow B
   \]

2. Eta for records:\(^6\)

   \[
   u : \Sigma\ A\ B \\
   u = (\text{proj}_1\ u, \text{proj}_2\ u) : \Sigma\ A\ B
   \]

\(^6\)Also known as surjective pairing for \(\Sigma\).
Definitional singleton types

Agda supports eta for all record types, not just \( \Sigma \)!
In particular, it has eta for the unit type:

\[
\text{record } \top : \text{Set where -- no fields} \\
\text{constructor } \texttt{tt} \\
\text{eta-unit } : (x y : \top) \rightarrow x \equiv y \\
\text{eta-unit } x y = \text{refl}
\]

Two distinct variables might be equal!

\( \Rightarrow \) To check if two terms are convertible, it does not suffice to compare their normal forms.
Encoding eta in Dedukti

1. Eta-expand everything when translating?

This is not stable under substitution:

\[
(a : A) \to N = A
\]

is not in eta-long form, but

\[a : A \to N\]

are.

2. Eta-reduce everything when translating?

This is not stable under substitution and:

\[(x : y \ x \ x) \to (z : z) = y, \to \beta x : z \ x, \to \eta z \]

but

\[x : y \ x \ x \neq, \to \eta z \neq, \to \eta\]

and

\[\_ : z \neq, \to \eta \neq, \to \eta\].
Encoding eta in Dedukti

1. Eta-expand everything when translating?
   *This is not stable under substitution:*

   \[
   \left(\lambda a : A. a\right)\{ \mathbb{N} \to \mathbb{N}/A \}
   \]

   is not in eta-long form, but \(\lambda a : A. a\) and \(\mathbb{N} \to \mathbb{N}\) are.
Encoding eta in Dedukti

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1. Eta-expand everything when translating?

This is not stable under substitution:

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is not in eta-long form, but \(\lambda a : A. a\) and \(N \to \mathbb{N}\) are.

2. Eta-reduce everything when translating?

This is not stable under substitution and \(\beta\):

\[(\lambda x. y \times x)\{((\lambda \_ . z)/y)\} \xrightarrow{\beta} \lambda x. z \times \xrightarrow{\eta} z\]

but \(\lambda x. y \times x \xrightarrow{\eta}\) and \(\lambda \_ . z \xrightarrow{\eta}\).
Encoding eta in Dedukti

3. Add eta-equality to the metatheory?
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*This only handles eta for the arrow type.*
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4. Use *eta-reduction* for record types?
Encoding eta in Dedukti

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   *This only handles eta for the arrow type.*

4. Use *eta-reduction* for record types?
   *This does not work for unit type, and needs non-linearity for the others:*
   
   \[ \text{mk\_pair (pi\_1 p) (pi\_2 p)} \rightarrow p \]
Encoding eta in Dedukti

3. Add eta-equality to the metatheory? 
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4. Use eta-reduction for record types? 
   *This does not work for unit type, and needs non-linearity for the others:* 
   \[
   \text{mk_pair (pi}_1 \ p \ \pi_2 \ p) \rightarrow p
   \]

5. Annotate terms with their types to be able to match them to eta expand? e.g. 
   \[
   \text{eta (arrow nat nat) f} \rightarrow x \Rightarrow f \ x
   \]
3. Add eta-equality to the metatheory? *This only handles eta for the arrow type.*

4. Use *eta-reduction* for record types? *This does not work for unit type, and needs non-linearity for the others:*
   \[
   \text{mk_pair} (\pi_1 p) (\pi_2 p) \rightarrow p
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5. Annotate terms with their types to be able to match them to eta expand? e.g.
   \[
   \text{eta} (\text{arrow} \text{nat} \text{nat}) f \rightarrow x \mapsto f x
   \]
   *We get bigger terms, and the other rules make the system non-confluent on pre-terms. Moreover, variables not translated as variables.*
Encoding eta in Dedukti

The next idea. Extend Dedukti with typed-directed rewrite rules.

Take inspiration from already existing works:

- Agda’s implementation of eta\(^7\)
- Andromeda 2’s extensionality rules\(^8\)

Or maybe there are still other unexplored options?

\(^7\)A. Bauer, A. Petković, An extensible equality checking algorithm for dependent type theories

\(^8\)https://agda.readthedocs.io/en/v2.6.2.2/language/record-types.html
Definitional irrelevance

Agda also supports definitional proof irrelevance\(^9\) for irrelevant functions and elements of Prop:

\[
\text{postulate} \\
P : \text{Prop} \\
f : P \rightarrow \mathbb{N}
\]

\[
P\text{-irrelevant} : (x \ y : P) \rightarrow f \ x \equiv f \ y \\
P\text{-irrelevant} \ x \ y = \text{refl}
\]

This causes very similar problems to eta for \(\top\), that also requires type-directed conversion to solve.

\(^9\)In the encoding of PVS we have a simpler form of proof irrelevance, which can be encoded in Dedukti.
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Summary

Many features of a dependently typed language can be encoded in Dedukti directly:

- Defined symbols are mapped to constants.
- Clauses are mapped to rewrite rules.

Other features require some more work:

- Erased constructor parameters need to be reconstructed.
- Universe levels require an equational theory.

Finally, other features we don’t yet know how to encode:

- Eta-equality for record types?
- Definitional proof irrelevance?
Future work

Like most translators, Agda2Dedukti is still a work in progress.

In the future, we would like to have:

- Compilation of clauses to elimination principles,
- A conservative encoding of universe polymorphism,
- An adequate and computational encoding of Agda,
- An encoding of eta-equality and irrelevance (probably requires extending Dedukti).

\[10\] For details, see Thiago’s talk about Adequate and Computational Encodings in Dedukti, at FSCD 2022
References


\(^{12}\)https://hal.inria.fr/hal-03343699