Safely Encoding B Proof Obligations in SMT-LIB

Final EuroProofNet Symposium
WG2: Workshop on Automated Reasoning and Proof Logging

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Introduction

B method

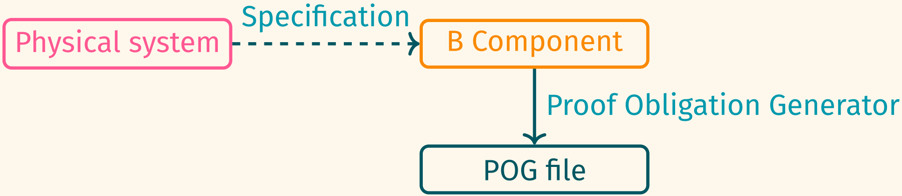
Physical system

Introduction

B method

Introduction

B method



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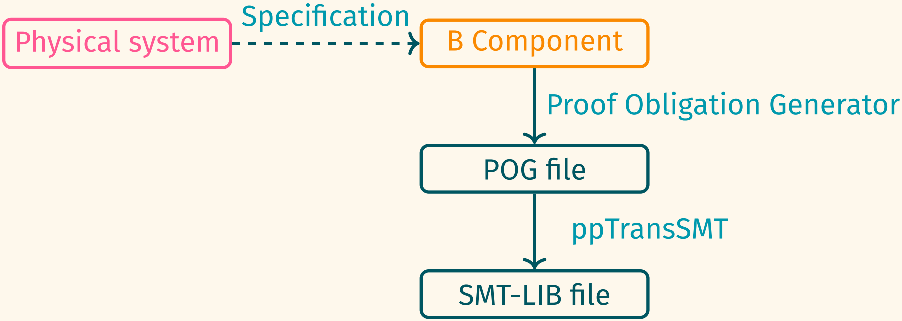

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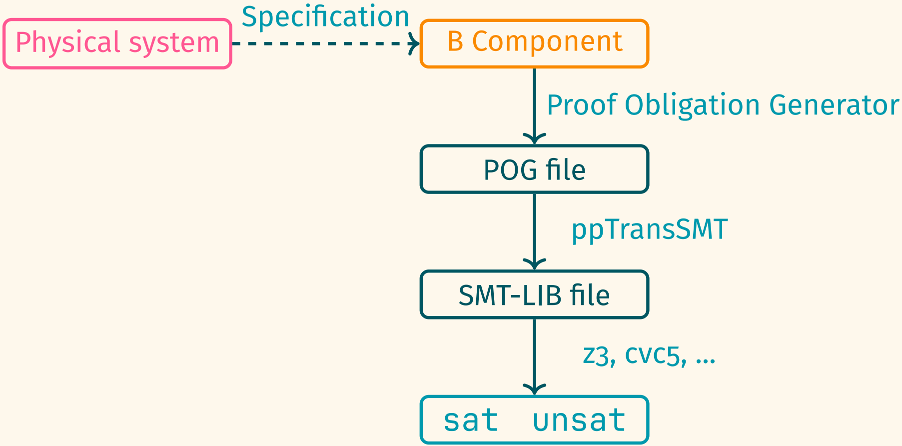

Introduction

SMT-LIB (up to v2.6)

- Standard input format for SMT solvers (e.g. z3, cvc5, veriT)
- Based on many-sorted **first-order logic**
- Comes with many theories (e.g. arrays, integer and real arithmetic)

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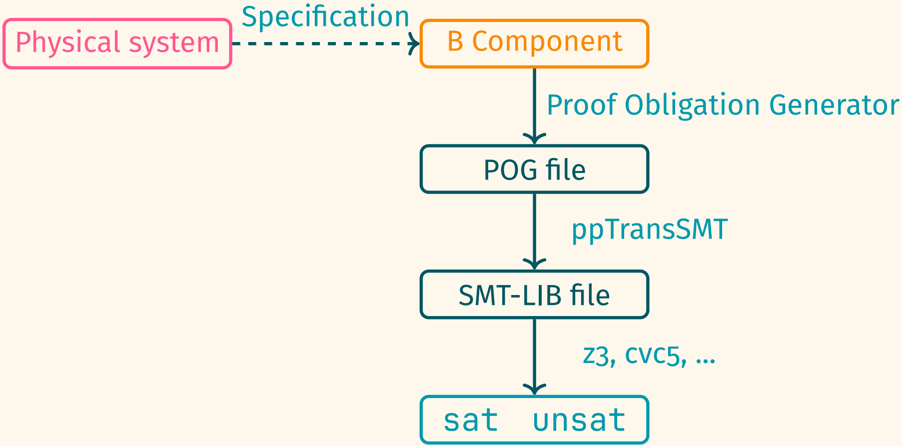
SMT-LIB v2.7

- Brings **higher-order constructs** through λ-abstractions
- Brings **higher-order types** through arrow type constructor
- Only supported by cvc5 yet

Introduction

B method



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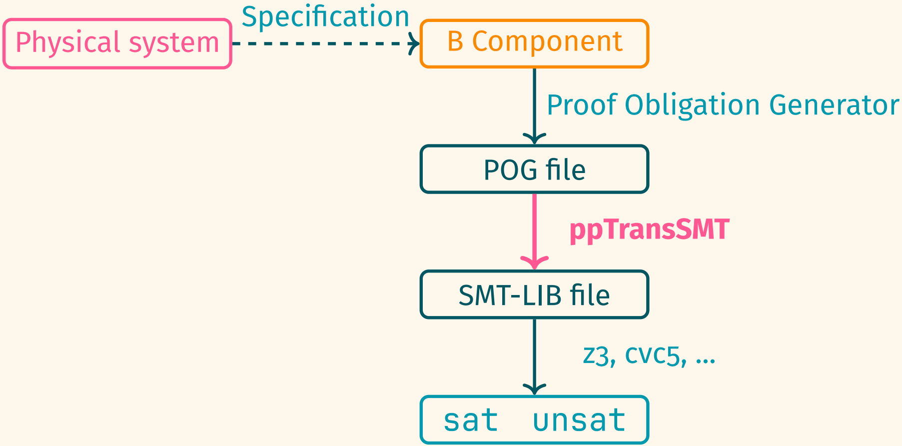

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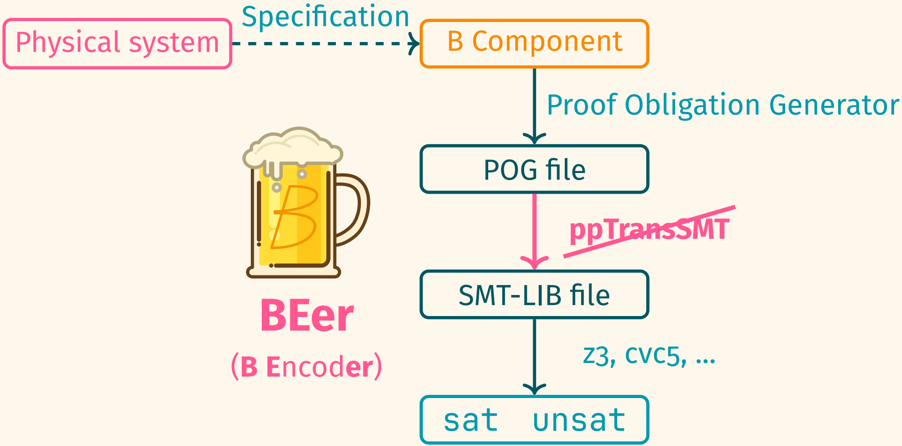

Introduction

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Introduction

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Architecture of

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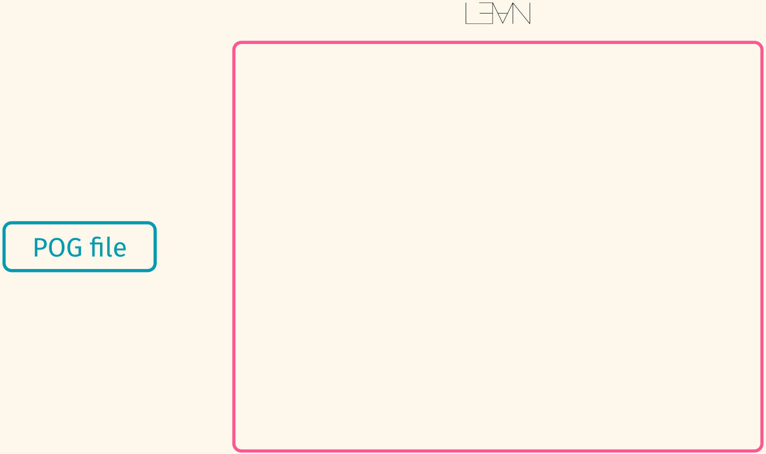

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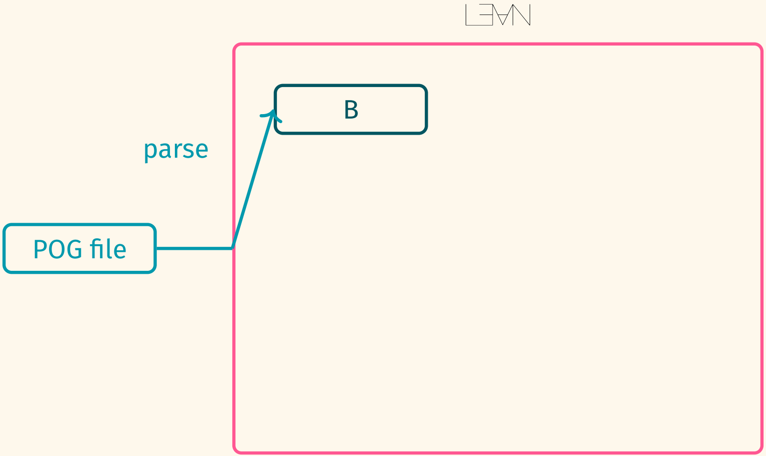

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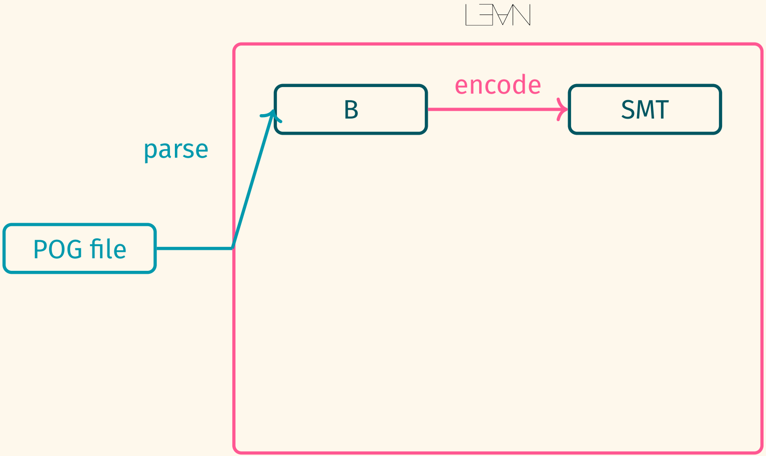

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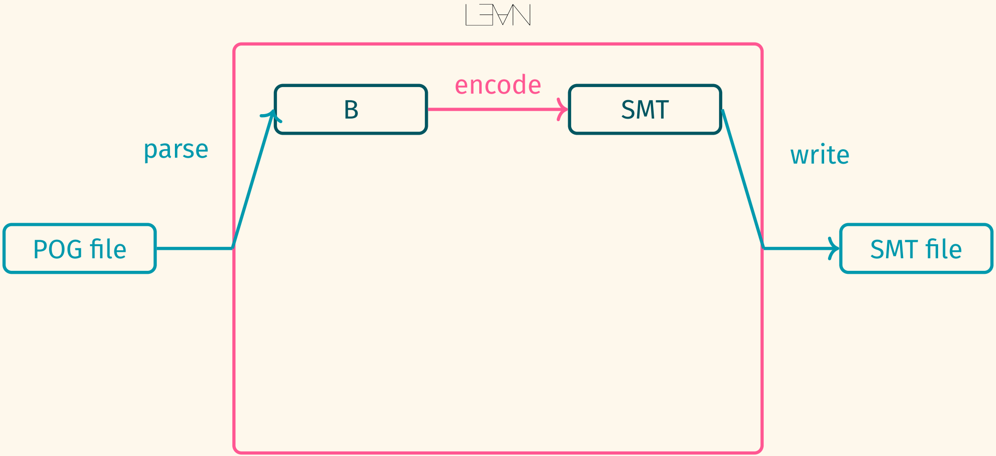

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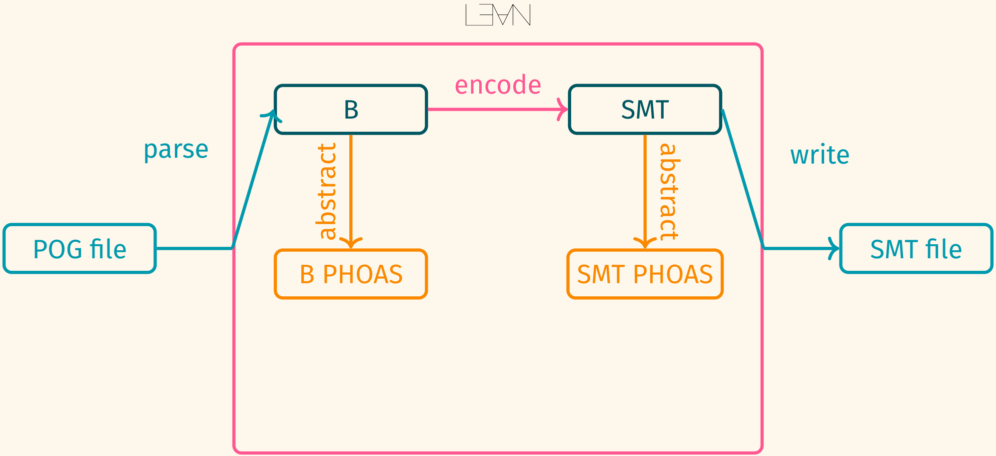

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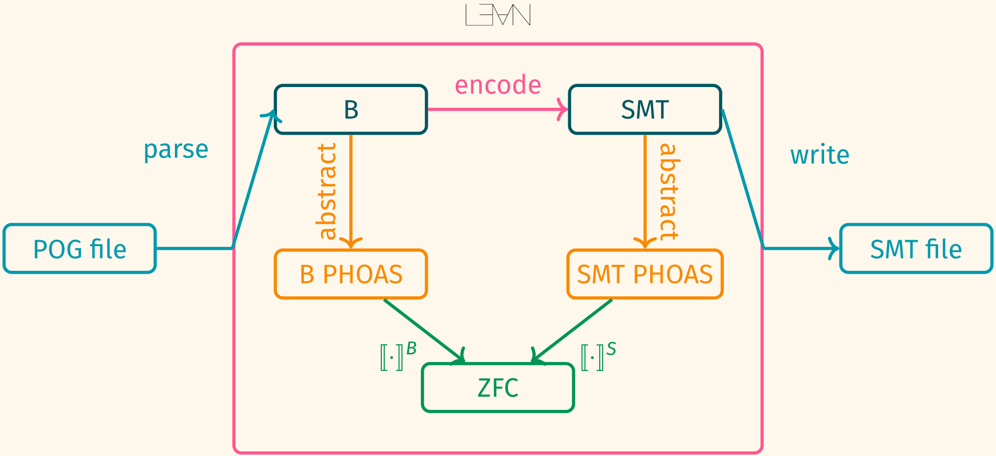

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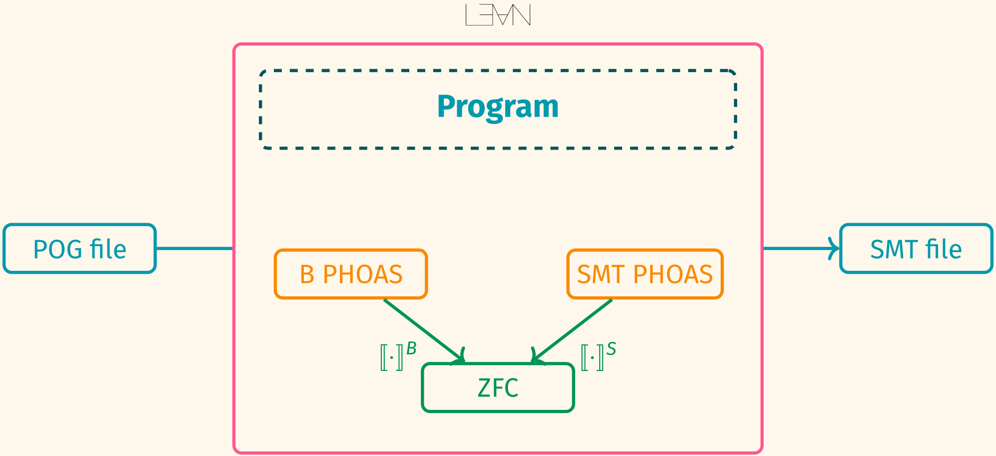

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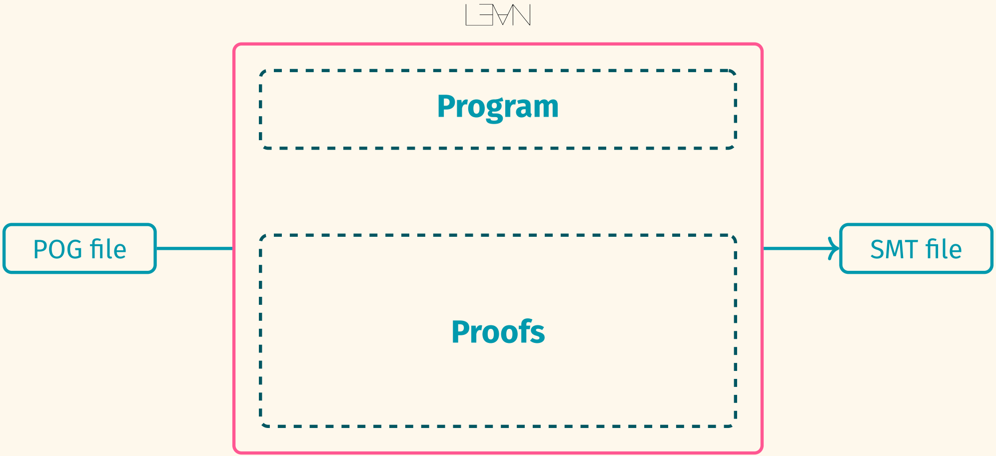

Overview of the encoding

(former) first-order encoding

(new) higher-order encoding 🍺

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(former) first-order encoding

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(declare-sort P 1)
(declare-sort C 2)
(declare-fun S () (P Int))
(declare-fun e1 () Int)
(declare-fun e2 () Int)
(declare-fun e3 () Int)
(assert (distinct e1 e2 e3))
(declare-fun ∈0 ((Int) (P Int)) Bool)

(assert (forall ((x Int)) (=
  (∈0 x S)
  (or (= x e1) (= x e2) (= x e3)))))
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(new) higher-order encoding 🍺

```
(declare-const e1 Int)
(declare-const e2 Int)
(declare-const e3 Int)
(assert (distinct e1 e2 e3))
(define-const S (→ Int Bool)
  (lambda ((x Int))
    (or (= x e1) (= x e2) (= x e3))))
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- Only expressions like $x \in S$ are encoded
- Functions are functional relations

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- Sets alone make sense; $x \in S$ is true by definition
- Functions are (sometimes) functions

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(former) first-order encoding

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$$f \subseteq A \times B$$

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(declare-datatype Option
  (par (T) ((some (the T)) (none))))
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(declare-sort P 1)
(declare-sort C 2)
(declare-const f (P (C τ_A τ_B)))
(declare-fun
  ∈_0 (τ_A τ_B (P (C τ_A τ_B))) Bool)
(assert
  (forall ((x τ_A) (y τ_B) (z τ_B))
    (⇒ (and (∈_0 x y f) (∈_0 x z f))
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+ specification that **dom** $f \subseteq A$ and **ran** $f \subseteq B$

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Let S be a set of elements of type τ.
The expression **finite** S can be encoded as follows:

$$
\begin{aligned}
&\exists N : \mathtt{int}, f : \tau \to \mathtt{int} \cdot \\
&\qquad \forall x : \tau, y : \tau, z : \mathtt{int} \cdot f(x) = z \land f(y) = z \Rightarrow x = y \qquad \land \\
&\qquad \forall x : \tau \cdot x \in S \Rightarrow 0 \leq f(x) \land f(x) < N
\end{aligned}
$$

Does this work?

```
MACHINE
  M
VARIABLES
  s0
INVARIANT
  s0 ⊆ NAT ∧
  s0 ∩ (ℤ \ ℕ) ∈ FIN(ℤ)
INITIALISATION
  s0 :∈ 𝒫(NAT)
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The following proof obligation is generated:

$$\mathtt{s0} \in \mathcal{P}(\mathbf{NAT}) \Rightarrow \mathtt{s0} \subseteq \mathbf{NAT} \land \mathtt{s0} \cap (\mathbb{Z} \setminus \mathbb{N}) \in \mathbf{FIN}(\mathbb{Z})$$

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✗ predicate prover from Atelier B

✗ CVC5 with ppTransSMT

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✗ predicate prover from Atelier B

✗ CVC5 with ppTransSMT

✓ CVC5 with 

Results

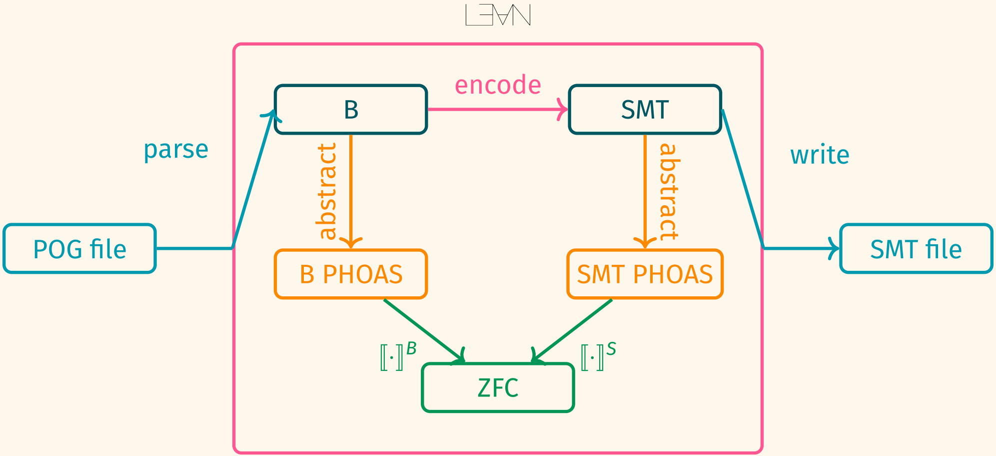
In the current state of we have:

ppTrans \	unsat	sat	unknown	Total
unsat	14,457	0	**431**	14,888
sat	1	1	5	7
unknown	**236**	5	472	713
Total	14,694	6	908	**15,608**

Benchmark specs:

- **681,285** POs in total
- Apple M2 (10 CPU cores, 24 GB RAM)
- cvc5 with incremental mode, MBQI enabled and 3s timeout per query

Architecture of 🍺



Roadmap

- ☐ Syntax
- ☐ Type system
- ☐ Semantics

Concrete terms

Concrete terms are **low-level** syntactic constructs used in the implementation of the encoding.

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```
inductive Term where
  | var (v : String) | int (n : Int) | bool (b : Bool) | x ↦ᴮ y
  -- arithmetic
  | x +ᴮ y | x -ᴮ y | x *ᴮ y | x ≤ᴮ y
  -- logic
  | x ∧ᴮ y | ¬ᴮ x | x =ᴮ y | ∀ᴮ (vs : List String) ∈ᴮ D ·P
  -- set operations
  | ℤ | 𝔹 | x ∈ᴮ y | 𝒫ᴮ S | S ×ᴮ T | S ∪ᴮ T | S ∩ᴮ T | |S|ᴮ
  | {(vs : List String) ∈ᴮ D | P}
  -- functions
  | app (f x : Term) | A ⇸ᴮ B | min (S : Term) | max (S : Term)
  | λᴮ (vs : List String) ∈ᴮ D | f
```

Concrete terms

Example

$$\lambda^{\mathsf{B}}[] \in^{\mathsf{B}} \mathit{int}\,2 \mid \mathit{var}\,\text{"}x\text{"} =^{\mathsf{B}} (\mathit{int}\,0 \wedge^{\mathsf{B}} \mathbb{B}) \qquad \mathit{int}\,0 +^{\mathsf{B}} \mathit{int}\,1 \in^{\mathsf{B}} \mathbb{Z}$$

$$\forall^{\mathsf{B}}[\text{"}x\text{"}, \text{"}y\text{"}] \in^{\mathsf{B}} \mathbb{Z} \times^{\mathsf{B}} \mathbb{Z} \cdot \mathit{var}\,\text{"}x\text{"} \leq^{\mathsf{B}} \mathit{var}\,\text{"}y\text{"} \qquad \mid \mathbb{Z} \mid^{\mathsf{B}} \in^{\mathsf{B}} \mathbb{Z}$$

are all syntactically valid terms.[1]

[1] Read with usual priorities.

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Some of these terms are type-correct, well-formed, or do not make sense.

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Abstract terms

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Remark

α-renaming and substitutions have to be handled explicitly:

$$(\lambda^{\mathsf{B}}[\text{"}x\text{"}] \in^{\mathsf{B}} \mathbb{Z} \mid var\ \text{"}x\text{"}) \neq (\lambda^{\mathsf{B}}[\text{"}y\text{"}] \in^{\mathsf{B}} \mathbb{Z} \mid var\ \text{"}y\text{"})$$

$$(\lambda^{\mathsf{B}}[\text{"}x\text{"}] \in^{\mathsf{B}} \mathbb{Z} \mid var\ \text{"}x\text{"} +^{\mathsf{B}} var\ \text{"}y\text{"})\ [\text{"}y\text{"} := \text{"}x\text{"}] = ???$$

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PHOAS

Variable management is delegated to the underlying formal system (Lean):

```
bbinder : List String → Term → Term → Term
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LEAN
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Example

Concrete term:

```
∀ᴮ ["x", "y"] ∈ᴮ ℤ ×ᴮ ℤ ·
  var "x" =ᴮ var "y"
```

Abstract term:

```
∀ᴮ ℤ ×ᴮ ℤ · fun x y ↦
  var x =ᴮ var y
```

Abstract terms

Abstraction function

Abstraction function: maps concrete terms to abstract terms under a renaming context

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Notation

> For any concrete term t and renaming context $\Delta : \mathtt{String} \to \mathtt{Option}\ \mathcal{V}$, the abstraction of t under Δ is denoted by $(\!| \ t \ |\!)_{\Delta}$.

Abstract terms

Abstraction function

Example

Let $Y \in \mathcal{V}$ and $\Delta := \{\text{"}y\text{"} \mapsto Y\}$. Consider the concrete term $\forall^{\mathsf{B}}[\text{"}x\text{"}] \in^{\mathsf{B}} \mathbb{Z} \cdot \mathit{var}\ \text{"}x\text{"} =^{\mathsf{B}} \mathit{var}\ \text{"}y\text{"}$.

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$$\begin{aligned} &(\!| \forall^{B}["x"] \in^{B} \mathbb{Z} \cdot \mathit{var}\ "x" =^{B} \mathit{var}\ "y" |\!)_{\Delta} \\ &= \forall^{B} (\!| \mathbb{Z} |\!)_{\Delta} \cdot \left(X \mapsto (\!| \mathit{var}\ "x" =^{B} \mathit{var}\ "y" |\!)_{\Delta["x" \mapsto X]} \right) \end{aligned}$$

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&= \forall^{\mathsf{B}} \llparenthesis\ \mathbb{Z}\ \rrparenthesis_{\Delta} \cdot \left(X \mapsto \llparenthesis\ \mathit{var}\ \text{"}x\text{"} =^{\mathsf{B}} \mathit{var}\ \text{"}y\text{"}\ \rrparenthesis_{\Delta[\text{"}x\text{"} \mapsto X]}\right) \\
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$$
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\end{aligned}
$$

Remark

In the actual implementation, Δ is required to contain all free variables of the term t being abstracted: $\mathrm{dom}(\Delta) \supseteq \mathrm{fv}(t)$

Roadmap

- ☐ Syntax
- ☐ Type system
- ☐ Semantics

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Type system

We introduce a basic type system for B, based on the following types:

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inductive BType where
| int | bool | set : BType → BType | _ ×ᴮ _ : BType → BType → BType
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which can be transformed into terms:

```
def BType.toTerm : BType → Term
| int     ⇒ ℤ
| bool    ⇒ 𝔹
| set α   ⇒ 𝒫ᴮ α.toTerm
| α ×ᴮ β  ⇒ α.toTerm ×ᴮ β.toTerm
```

Typing rules

Type contexts are defined as follows:

LEAN
```
abbrev TypeContext :=
  AList fun _ : String ↦ BType
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abbrev PHOAS.TypeContext 𝒱 :=
  𝒱 → Option BType
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together with an abstraction function:

```lean
noncomputable def TypeContext.abstract {𝒱}
  (Δ : String → Option 𝒱) Γ : PHOAS.TypeContext 𝒱 := fun x : 𝒱 ↦
    if h : ∃ v ∈ Γ, Δ v = some x  then Γ.lookup (choose h)
    else none
```

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Similarly: $\mathit{add}_{\mathrm{I}}$, $\mathit{mul}_{\mathrm{I}}$, $\mathit{sub}_{\mathrm{I}}$, le_{I}, $\mathit{not}_{\mathrm{I}}$, eq_{I}, $\mathbb{Z}_{\mathrm{I}}$, $\mathbb{B}_{\mathrm{I}}$, $\mathit{mem}_{\mathrm{I}}$, $\mathit{pow}_{\mathrm{I}}$, $\mathit{cprod}_{\mathrm{I}}$, $\mathit{union}_{\mathrm{I}}$, $\mathit{inter}_{\mathrm{I}}$, $\mathit{card}_{\mathrm{I}}$, $\mathit{min}_{\mathrm{I}}$, $\mathit{max}_{\mathrm{I}}$.

Typing rules

Concrete lambda typing rule

Let $n \in \mathbb{N}^*$, Γ a concrete type context, $(v_i)_{i<n}$ concrete variables, $(\alpha_i)_{i<n}$ and ξ B types, $(D_i)_{i<n}$ and f concrete terms. Assume the following:

$$\forall i < n, v_i \notin \Gamma \,; \quad \forall i < n, \Gamma \vdash D_i : \mathit{set}\ \alpha_i \,; \quad \Gamma[v_i \coloneqq \alpha_i]_{i<n} \vdash f : \xi$$

Then, the following typing judgment holds:

$$\Gamma \vdash \lambda^{\mathrm{B}} v_1, \ldots, v_n \in^{\mathrm{B}} D_1 \times^{\mathrm{B}} \ldots \times^{\mathrm{B}} D_n \cdot f : \mathit{set}\ (\alpha_1 \times^{\mathrm{B}} \ldots \times^{\mathrm{B}} \alpha_n \times^{\mathrm{B}} \xi)$$

Typing rules

Abstract lambda typing rule

Let $n \in \mathbb{N}^*$, Γ an abstract type context, $(\alpha_i)_{i<n}$ and ξ B types, $(D_i)_{i<n}$ abstract terms and f an abstract motive. Assume the following:

$$\forall i < n,\ \Gamma \vdash D_i\ :\ set\ \alpha_i\ ; \quad \forall (v_i)_{i<n},\ \Gamma[v_i := \alpha_i]_{i<n} \vdash f\ ((v_i)_{i<n})\ :\ \xi$$

Then, the following typing judgment holds:

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Theorem (Weakening)

Assume that $\Gamma \vdash t : \tau$. *Let* v *be a variable such that* $v \notin \Gamma$. *Then,*

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Theorem (Determinism)

$$\Gamma \vdash t : \tau \to \Gamma \vdash t : \sigma \to \tau = \sigma$$

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ZFC

Interpreting the B syntax in ZFC set theory:

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where

$$\texttt{Dom} := \sum_{x,\tau} x \in [\![\tau]\!]^{Z}$$

$$\begin{cases} [\![int]\!]^{Z} & := \mathbb{Z}^{Z} \\ [\![bool]\!]^{Z} & := \mathbb{B}^{Z} \\ [\![set\ \alpha]\!]^{Z} & := \mathcal{P}^{Z}([\![\alpha]\!]^{Z}) \\ [\![\alpha \times^{B} \beta]\!]^{Z} & := [\![\alpha]\!]^{Z} \times^{Z} [\![\beta]\!]^{Z} \end{cases}$$

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Simple cases

Let's consider $\mapsto^{\mathsf{B}}$ only; other binary cases are similar. Let x and y be two abstract terms such that:

$$[\![x]\!]^{\mathsf{B}} = \langle X, \alpha, \mathfrak{pf}_X^{\alpha} \rangle \quad \text{and} \quad [\![y]\!]^{\mathsf{B}} = \langle Y, \beta, \mathfrak{pf}_Y^{\beta} \rangle$$

The denotation is then constructed as follows:

$$[\![x \mapsto^{\mathsf{B}} y]\!]^{\mathsf{B}} := \langle (X, Y)^{\mathsf{Z}}, \alpha \times^{\mathsf{B}} \beta, \mathfrak{pf}_{(X,Y)^{\mathsf{Z}}}^{\alpha \times^{\mathsf{B}} \beta} \rangle$$

where

$$\mathfrak{pf}_{(X,Y)^{\mathsf{Z}}}^{\alpha \times^{\mathsf{B}} \beta} \dashv \underbrace{X \in [\![\alpha]\!]^{\mathsf{Z}}}_{\mathfrak{pf}_X^{\alpha}} \wedge \underbrace{Y \in [\![\beta]\!]^{\mathsf{Z}}}_{\mathfrak{pf}_Y^{\beta}}$$

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Forall quantifier

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We define the following function, for any ZFC set z:

$$\mathscr{P}(z) := \begin{cases} P_z & \text{if } \exists (x_i)_{1 \le i \le n}, \begin{cases} z = (x_1, \ldots, x_n)^{\mathsf{Z}} \wedge z \in [\![\tau]\!]^{\mathsf{Z}} \\ \left[\!\!\left[P\ \left(\mathit{var}\ \langle x_i, \tau_i, \mathfrak{pf}_{x_i}^{\tau_i} \rangle\right)_{1 \le i \le n} \right]\!\!\right]^{\mathsf{B}} = \langle P_z, -, - \rangle \end{cases} \\ \bot^{\mathsf{Z}} & \text{otherwise} \end{cases}$$

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Let $D\colon \texttt{Term Dom}$ and $P\colon (\texttt{Fin}\ n \to \texttt{Dom}) \to \texttt{Term Dom}$. Assume that

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At last, we define the following denotation:

$$[\![\forall^{\mathsf{B}} D \cdot P]\!]^{\mathsf{B}} \coloneqq \left\langle \bigwedge_{x\in\mathcal{D}}^{\mathsf{Z}} \mathscr{P}\ x,\ \mathit{bool},\ \mathfrak{pf}_{\forall^{\mathsf{B}} D\cdot P}^{\mathit{bool}} \right\rangle \quad \text{where} \quad \mathfrak{pf}_{\forall^{\mathsf{B}} D\cdot P}^{\mathit{bool}}\colon \bigwedge_{x\in\mathcal{D}}^{\mathsf{Z}} \mathscr{P}\ x \in [\![\mathit{bool}]\!]^{\mathsf{Z}}$$

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(easy proof)

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- ☑ Type system
- ☐ Semantics

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Theorem (Type correctness of the denotation)

Assume that $\Gamma \vdash t : \tau$. *Then,*

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Theorem (Partial correctness of the simplifier)

Assume that $\Gamma \vdash t : \tau$. *Let* Δ *be a renaming context. Then,*

$$[\![(\!| t |\!)_{\Delta}]\!]^{\mathsf{B}} = \langle T, \tau, \mathfrak{pf}^{\tau}_{T} \rangle \rightarrow [\![(\!| \mathsf{simplifier}(t) |\!)_{\Delta}]\!]^{\mathsf{B}} = \langle T, \tau, \mathfrak{pf}^{\tau}_{T} \rangle$$

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Remark

Total correctness does not hold.

Conclusion

Contributions:

- **Higher-order encoding** leveraging recent advances in SMT solvers
- **Formal semantics** for subsets of B proof obligations and SMT-LIB

Current/future work:

- **Correctness** of the encoding