Yalep: An environment for learning proof in high-school

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Europroofnet-MCLP, September 2025

Funded by the European Union

- Learning proof is hard !
- Why not teaching with proof assistants ?
  - Hands-on approach
  - Students receive instant and frequent feedback
  - Helps them realize that a proof could be mechanically verified.
- PhD ! (w/ Julien Narboux, Laure Gonnord)
  $\longrightarrow$ explore some questions raised by this educational setting

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- Many experiments of teaching w/ many different PAs[2]

  using Rocq (Kerjean, Mayero, Rousselin), Deaduction (Leroux), LeanVerbose (Massot), Edukera & Lean proof term (Tran Minh), Deaduction & LeanVerbose (Bartzia, Boutry, Narboux), Edukera (Modeste), Coq Waterproof (Wemmenhove), Proof Buddy (Karsten), . . .
- APPAM: didacticians, mathematicians and computer scientists
- An a priori analysis[3]

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The idea is as old as PA themselves[1]

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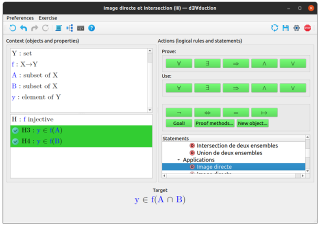

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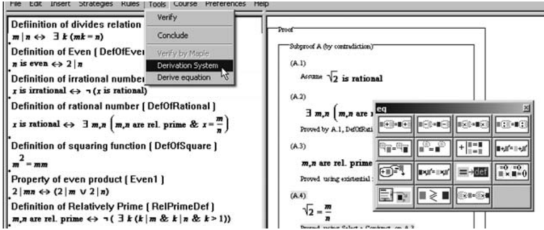
- GUI proofs may not be so instructive[3].
- Our focus : language
- Rare tools designed for High school (except for Geometry)



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- What is specific to a high school proof ?
  - rather short, informal proofs
  - quantifiers, logical connectors in plain text
- What is specific to the high school context ?
  - students' supposed background: PA (none) ; CS (none) ; proof (tiny)
  - teachers' supposed background: PA (none) ; CS (none) ; logics (usually little)
  - no additional training
- what is specific to the French high school curriculum
  - recommends gradually guiding students toward the truth through proof.
  - some example proofs ; ex: $\sqrt{2} \notin \mathbb{Q}$
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High school proofs?

Prove : $\forall n \in \mathbb{Z}, n(n+1)$ is even.

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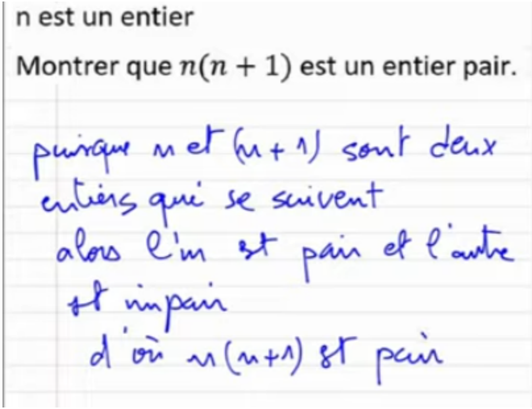
n est un entier

Montrer que $n(n+1)$ est un entier pair.

puisque n et (n+1) sont deux
entiers qui se suivent
alors l'un est pair et l'autre
est impair
d'où n(n+1) est pair

Since n and $n+1$ are two consecutive integers, then one is even and the other is odd, so $n(n+1)$ is even.

High school proofs?

Prove : $\forall n \in \mathbb{Z}, n(n+1)$ is even.

Deux entiers consécutifs s'écrivent, par exemple, sous la forme n et $n+1$.

- Si n est pair, il existe alors un entier relatif k tel que $n = 2k$.
  Ainsi $n(n+1) = 2k(n+1)$ est pair.
- Si n est impair, il existe alors un entier relatif k tel que $n = 2k+1$.
  Par conséquent $n+1 = 2k+1+1 = 2k+2 = 2(k+1)$.
  Ainsi $n(n+1) = n \times 2(k+1)$ est pair.

Two consecutive integers are written, for instance, in the form n and $n+1$.

- If n is even, then there exists an integer k such that $n = 2k$. Therefore, $n(n+1) = 2k(n+1)$ is even.
- If n is odd, then there exists an integer k such that $n = 2k+1$. As a result, $n+1 = 2k+1+1 = 2k+2 = 2(k+1)$. Thus, $n(n+1) = n \times 2(k+1)$ is even.

Prove : $\forall n \in \mathbb{Z}, n(n+1)$ is even.

```
theorem product_even (n a:ℤ) :  Even n → Even (n*a)
  | ⟨k,hk⟩ => by use k*a ; rw[← right_distrib,hk]

theorem successor_odd (n  :ℤ) : Odd n → Even (n+1)
  | ⟨k,hk⟩ => by use k+1 ; rw[hk] ; ring_nf

example (h: ∀n:ℤ, Even n ∨ Odd n) : ∀ n:ℤ, Even (n*(n+1)) := by
  intro n
  cases (h n)
  apply product_even
  assumption
  rw [mul_comm]
  apply product_even
  apply successor_odd
  assumption
```

A pedagogical progression as a guideline

proof construct	*example of statement to prove*
new facts - goal	$\vdash 81 = 2 * 40 + 1$
Intro $\exists$	$\vdash$ 81 odd
Elim $\exists$	$n \in \mathbb{Z}, n$ odd $\vdash n + 1$ even $n \in \mathbb{Z}, n$ odd $\vdash n^2$ odd
Intro $\Longrightarrow$	n integer $\vdash n$ even $\Longrightarrow n^2$ even
Intro $\forall$	$\forall n \in \mathbb{Z}, \forall b \in \mathbb{Z}, n$ even $\Longrightarrow nb$ even
Intro $\vee$	$\vdash$ (0 even) $\vee$ (0 odd)
Elim $\vee$ (by cases)	$\forall n \in \mathbb{Z}$, (n even) $\vee$ (n odd) $\vdash \forall n \in \mathbb{Z}, n(n+1)$ even
Intro $\neg$	$\vdash \neg$(1 even) $\vdash \forall n \in \mathbb{Z}, \neg(n$ odd $\wedge$ n even)
induction	$\vdash \forall n \in \mathbb{N}, n \leqslant n^2$. $\vdash \forall n \in \mathbb{N}, n$ odd $\vee$ n even $\vdash \forall n \in \mathbb{Z}, n$ odd $\vee$ n even
Intro of $\Longleftrightarrow$	$\vdash \forall n \in \mathbb{Z}, n$ odd $\Longleftrightarrow \neg(n$ even)
contrapositive	$\vdash \forall n \in \mathbb{Z}, n^2$ even $\Longrightarrow n$ even
Synthesis	$\sqrt{2} \notin \mathbb{Q}$

Progression

- theme: parity
- leads to $\sqrt{2} \notin \mathbb{Q}$
- explores all logical connectors

Milesones

- Dec 2024 w/ 9th-graders
- June 2025 w/ 10th-grade

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"Yet Another Learning Environment for Proof" (YALEP) - A first taste

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Theorem nn1even "7. Using (... or ...) : proof by cases"
    Assumptions: (for all integer n, n is even or n is odd)
    Conclusion: for all integer n, n*(n+1) is even
Proof
    let n be an integer
    ◆ n is even or n is odd
    ◆ if n is even then n*(n+1) is even
      proof
        assume n is even
        ◆ n*(n+1) is even by product_even
      □
    ◆ if n is odd then n*(n+1) is even
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Proof frame

Fact (◆)

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Proof frame
Fact ◆)
Sub-proof

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Proof frame
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Sub-proof
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Proof frame
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Syntactic sugar

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Proof frame
Fact ◆)
Sub-proof
Silent
Short
Syntactic sugar
Reduced vocabulary

What language is appropriate to teach proof to high-school students?

Under the constraint : *without additional training*, the language should be:

- Readable by students (without proof state display)
  - $\longrightarrow$ imitate pen and paper writing
  - $\longrightarrow$ declarative
- Writable by students
  - $\longrightarrow$ reduced vocabulary and set of proof constructs

Hope : practicing Yalep will help to transfer *proving skills* to pen and paper proof activity (and not copying proof scripts!).

Why does our need diverge from NL parsing?

(Diproche Carl, Lorenzen, and Schmitz 2022)

```
Let x be an integer.
Prove: If x is even, then 2-3x is even.
  Proof:
    Let x be even.
    Then, there is an integer k such that x=2k.
    Let k be an integer with x=2k.
    Then we have 2-3x=2-3(2k)=2(1-3k).
    Hence 2-3x is even.
  qed.
```

NL parsing [Naproche Koepke 2019]:

- has different objectives (parsing textbooks)
- to accept any NL phrase, has to cope with implicit and ambiguities

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- to accept any NL phrase, has to cope with implicit and ambiguities

- Proof λ-term : internal representation of proof in a proof assistant based on typed λ-calculus.

```
example (x:integer) : x is even ⇒ 2-3*x is even :=
 λ assumption1 ↦
  Exists.elim assumption1 λ k fact1 ↦
   Exists.intro (1-3*k) <|
    Eq.trans (congrArg (2-3*·) fact1)                    <|
    Eq.trans (congrArg (2-·)   $ (mul_assoc 3 2 k).symm)  <|
    Eq.trans (congrArg (2-·*k) $ (mul_comm 3 2))          <|
    Eq.trans (congrArg (2-·)   $ (mul_assoc 2 3 k))       <|
    Eq.trans (congrArg (· - 2*(3*k)) (rfl: (2:Int)=2*1))
      (mul_sub (2:Int) 1 (3*k)).symm
```

```
example (x:integer) :
  x is even ⇒ 2-3*x is even :=
by
  intro assumption1
  obtain ⟨k,fact1⟩ := assumption1
  use (1-3*k)
  rw[fact1]
  ring_nf
```

Proof state

```
1 goal
x : integer
⊢ x is even ⇒ 2 - 3 * x is even
```

- Procedural proof script : sequence of ”tactics” building a proof term
- Focus on proof actions
- Not meant to be human-readable: needs proof state display to replay the proof

Different proof styles : procedural proof script (Lean)

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Proof state

```
1 goal
case intro
x k : integer
fact1 : x = 2 * k
⊢ 2 - 3 * x is even
```

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Proof state

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  obtain ⟨k,fact1⟩ := assumption1
  use (1-3*k)
  rw[fact1]
  ring_nf
```

Proof state

```
No goals
```

- Procedural proof script : sequence of ”tactics” building a proof term
- Focus on proof actions
- Not meant to be human-readable: needs proof state display to replay the proof

Different proof styles : declarative proof script (Lean)

```
example (x:integer) : x is even ⇒ 2-3*x is even :=
by
  intro (_ : x is even)
  obtain ⟨k,fact1 : x = 2*k⟩ := by assumption

  have : 2-3*x = 2*(1-3*k) := calc
                              2-3*x = 2-3*(2*k)   := by rw[fact1]
                              _      = 2*(1-3*k ) := by ring_nf

  have : ∃ u:ℤ, 2-3*x = 2*u := by use (1-3*k)
  have : 2-3*x is even       := by assumption

  assumption
```

- Declarative proof script : list of claims with corresponding subproof
- Focus on statement : closer to pen and paper practice

Different proof styles : Controlled Natural Language for education

```
Exercise "if x is even then 2-3*x is even"
  Given: (x:integer)
  Conclusion: x is even ⇒ 2-3*x is even
Proof:
  Assume that assumption1: x is even
  Since x is even we get k such that fact1: x = 2*k
  Let's prove that (1-3*k) works
  Calc
    2-3*x = 2-3*(2*k)  by We rewrite using fact1
    _     = 2*(1-3*k ) by computation
QED
```

Lean Verbose (Massot 2024)

```
Goal 2 is the infimum of [2, 5).
Proof.
We need to show that (2 is a _lower bound_ for [2, 5)
  ∧ (∀ l ∈ ℝ, l is a _lower bound_ for [2, 5) ⇒ l ⩽ 2)).
We show both statements.
  - We need to show that (2 is a lower bound for [2, 5)).
    We need to show that (∀ c ∈ [2, 5), 2 ⩽ c).
    Take c ∈ [2, 5).
    We conclude that (2 ⩽ c).
  - We need to show that
      (∀ l ∈ ℝ, l is a lower bound for [2, 5) ⇒ l ⩽ 2).
    Take l ∈ ℝ. Assume that (l is a lower bound for [2, 5)).
    We conclude that (l ⩽ 2).
Qed.
```

Coq Waterproof (Wemmenhove et al. 2024)

- Original tactics are renamed or redefined to better fit vernacular language
- Can mix declarative and procedural
- Goal : improve transfer to pen and paper proof
- Possible bias : it is strict programming, not natural language
- Remedy : extend vocabulary (policy?)

```
Theorem "if x is even then 2-3*x is even"
  Assumptions: (x is integer)
  Conclusion: if x is even then 2-3*x is even
Proof
  assume x is even
  ◆ there exists an integer k such that x=2*k
  obtain such k
  ◎ 2-3*x = 2-3*(2*k)
    _     = 2*(1-3*k)
  ◆ 2-3*x is even
□
```

dialect	*keep*	*abandon*
λ-term	structure, proof object	requires to learn functional programming
procedural	proof state	needs replaying with proof state to be understood
declarative	statements	proofs (at least if too detailed)
CNL	syntactic sugar	vocabulary extensibly large to imitate NL parsing

Towards a minimal language for high-school proofs

Key ingredients for minimality

1. No redundancy : `let` ~~fix~~ ~~take~~ / `assume` ~~suppose~~ ~~let~~ / `obtain` ~~fix~~ ~~let~~ ...
2. Avoid commands : instead of unfold, rewrite, compute : state a new fact
3. Implicit proof actions eliminate the need for specific tactics

   $\longrightarrow$ Approach inspired from coherent logic (Stojanovic et al. 2014)

Connector	Introduction	Elimination
`P and Q`	(silent)	(silent)
`P or Q`	(silent)	(silent)
$P \Longrightarrow Q$	`assume P`	(silent)
$\forall x \in E, P(x)$	`let x ∈ E`	(silent)
$\exists x \in E, P(x)$	(silent)	`obtain such x`
$P \Longleftrightarrow Q$	(silent)	(silent)

4. Favor forward chaining (see next slide...)

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Connector	Introduction	Elimination
`P and Q`	(silent)	(silent)
`P or Q`	(silent)	(silent)
$P \implies Q$	`assume P`	(silent)
$\forall x \in E, P(x)$	`let x` $\in$ `E`	(silent)
$\exists x \in E, P(x)$	(silent)	`obtain such x`
$P \iff Q$	(silent)	(silent)

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$P \Longleftrightarrow Q$	(silent)	(silent)

4. Favor forward chaining (see next slide...)

Favoring forward chaining

Forward chaining	Backward chaining

let n be an integer
◆ n is even or n is odd
◆ if n is even then n*(n+1) is even
  proof
    assume n is even
    ◆ n*(n+1) is even by product_even
  □
◆ if n is odd then n*(n+1) is even
  proof
    assume n is odd
    ◆ n+1 is even
    ◆ n*(n+1) is even by product_even
  □
◆ n*(n+1) is even
	Fact f1: n is even ∨ n is odd
	  from every_integer_is_even_or_odd
	We discuss depending on whether
	  n is even or n is odd
	Assume that h1: n is even
	We conclude by product_even
	  applied to n and (n+1) and h1
	Assume that h2: n is odd
	We rewrite using mul_comm
	We apply product_even
	We conclude by successor_odd
	  applied to n and h2

• Forces students to explicit their goals • Relaxes constraint on order • Requires less vocabulary	• Automatic opening of several goals • Order of sub-goals imposed • Appropriate vocabulary needed

Favoring forward chaining

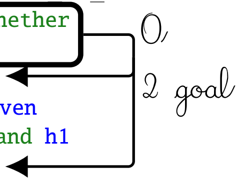
Forward chaining

```
let n be an integer
◆ n is even or n is odd
◆ if n is even then n*(n+1) is even
  proof
    assume n is even
    ◆ n*(n+1) is even by product_even
  □
◆ if n is odd then n*(n+1) is even
  proof
    assume n is odd
    ◆ n+1 is even
    ◆ n*(n+1) is even by product_even
  □
◆ n*(n+1) is even
```

- Forces students to explicit their goals
- Relaxes constraint on order
- Requires less vocabulary

Backward chaining

```
Fact f1: n is even ∨ n is odd
  from every_integer_is_even_or_odd
We discuss depending on whether
  n is even or n is odd
Assume that h1: n is even
We conclude by  product_even
  applied to n and (n+1) and h1
Assume that h2: n is odd
We rewrite using mul_comm
We apply product_even
We conclude by successor_odd
  applied to n and h2
```



- Automatic opening of several goals
- Order of sub-goals imposed
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Favoring forward chaining

Forward chaining	Backward chaining

Forward chaining

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let n be an integer
◆ n is even or n is odd
◆ if n is even then n*(n+1) is even
  proof
    assume n is even
    ◆ n*(n+1) is even by product_even
  □
◆ if n is odd then n*(n+1) is even
  proof
    assume n is odd
    ◆ n+1 is even
    ◆ n*(n+1) is even by product_even
  □
◆ n*(n+1) is even
```

fact

Backward chaining

```
Fact f1: n is even ∨ n is odd
  from every_integer_is_even_or_odd
We discuss depending on whether
  n is even or n is odd
Assume that h1: n is even
We conclude by  product_even
  applied to n and (n+1) and h1
Assume that h2: n is odd
We rewrite using mul_comm
We apply product_even
We conclude by successor_odd
  applied to n and h2
```

2 goal

- Forces students to explicit their goals
- Relaxes constraint on order
- Requires less vocabulary

- Automatic opening of several goals
- Order of sub-goals imposed
- Appropriate vocabulary needed

- Proof assistants already provide real time display of proof state (context, goal) and correctness: 1st level of feedback
- Lean provides
  - Flexible parser and pretty printer (Massot 2024)
  - Comprehensive, unified and community maintained Math Library (The Mathlib Community 2020)
  - Powerful tactic automation
  - Lean module allowing to interact with React/JavaScript widgets (Nawrocki, Ayers, and Ebner 2023)
  - LeanWeb interface enables to run Lean in a web browser[2]

[2] https://github.com/leanprover-community/lean4web

Hiding types : a proof of $\sqrt{2} \notin \mathbb{Q}$ in bare Lean

```
def Rationals : Set ℝ := {x:ℝ | ∃ p:ℤ, ∃ q:ℕ, q ≠ 0 ∧ x=(p:ℝ)/(q:ℝ) ∧ ¬(p is even ∧ q is even)}
notation (priority := high) "ℚ" => Rationals
example : √2 ∉ ℚ := by
    intro (assumption1: √2 ∈ ℚ)
    let ⟨p,q,(f03: q ≠ 0),(f04: √2 = p/q),(f05 : ¬ (p is even ∧ q is even))⟩ := assumption1

    -- ensure that all computations are done with (p:ℝ) and (q:ℝ)
    have f06 : (p:ℝ)^2 = (2:ℝ)*(q:ℝ)^2  :=
      calc
        (p:ℝ)^2 = ((p:ℝ)^2/(q:ℝ)^2)*q^2      := by field_simp
        _       = (((p:ℝ)/(q:ℝ))^2) * q^2    := by ring_nf
        _       = ((√2)^2) * (q:ℝ)^2          := by rw [<√2 = (p:ℝ)/(q:ℝ)>]
        _       = (2:ℝ)*(q:ℝ)^2               := by norm_num

    -- back to (p:ℤ) and (q:ℤ) (using injectivity of coercion morphism)
    have f07 : p^2 = 2*q^2      := by rify ; rw[f06]
    have f08 : (p^2) is even    := by use q^2
    have f09 : p is even        := by apply n2_even_implies_n_even ; assumption

    let ⟨(k:ℤ), (f10: p = 2*k)⟩ := f09

    have f11 :  (2*k)^2    = 2*q^2      := by rw [← <p = 2*k> , <p^2 = 2*q^2>]
    have f13 :  2*k^2      = q^2        := by linarith
    have f14 :  (q^2) is even           := by use k^2 ; apply Eq.symm ; assumption
    have f15 : q is even                := by apply n2_even_implies_n_even ; assumption
    have f16 : p is even                := by apply n2_even_implies_n_even ; assumption
    have f17 : (p is even ∧ q is even) := by constructor <;> assumption
    contradiction
```

```
-- ...obtain  (p:ℤ) and (q:ℕ) such that f04: √2 = p/q
-- ... and p and q not both even

-- ensure that all computations are done with (p:ℝ) and (q:ℝ)
have f06 : (p:ℝ)^2 = (2:ℝ)*(q:ℝ)^2  :=
  calc
    (p:ℝ)^2 = ((p:ℝ)^2/(q:ℝ)^2)*q^2     := by field_simp
    _       = (((p:ℝ)/(q:ℝ))^2) * q^2   := by ring_nf
    _       = ((√2)^2) * (q:ℝ)^2         := by rw [f04]
    _       = (2:ℝ)*(q:ℝ)^2              := by norm_num

-- back to (p:ℤ) and (q:ℤ) (using injectivity of coercion morphism)
have f07 : p^2 = 2*q^2      := by rify ; rw[f06]
-- ...
```

- Bertot and Portet 2025 advocate for a unique number type.
  $\longrightarrow$ They formalize $\mathbb{N}$ as an inductive predicate over the `Real` type.
- Yalep integrates the same idea with a simpler formalization :

Given	a reference type a type to be represented an injective coercion morphism	T : Type E : Type $c : E \to T$
we expose	the image of E under c : $c\langle E\rangle := \{x : T \mid \exists y : E, x = c(y)\}$	
we hide	the underlying types : E , T	

- When $T =$ `Real`, this applies similarly for $E \in \{$`Nat`, `Int`, `Rat`$\}$.
- Automation should prove stability statements like
  $\forall x, \forall y, x \in c\langle E\rangle \wedge y \in c\langle E\rangle \Rightarrow x + y \in c\langle E\rangle$.

Hiding types : a proof of $\sqrt{2} \notin \mathbb{Q}$ in Yalep

```
Theorem sqrt_2_is_irrational "√2 ∉ ℚ"
 Conclusion: √2 is not rational
Proof
 assume that √2 is rational
 ◆ there exists an integer p such that
    there exists a natural number q
     such that q≠ 0 and √2 = p/q and
     the statement (p is even and q is
    even) is false

 obtain such p
 obtain such q

 ◎ p^2 = (p/q)^2 * q^2
   _   = (√2)^2  * q^2
   _   =  2      * q^2
```

```
 ◆ p^2 is even
 ◆ p is even by n2_even_implies_n_even
 obtain an integer k such that p = 2*k
 ◎ q^2 = (2*q^2)/2
   _   = p^2 / 2
   _   = (2*k)^2 /2
   _   = 2 * k^2
 ◆ q^2 is even
 ◆ q is even by n2_even_implies_n_even
 ◆ Absurd
□
```

```
let's define the function f
 from [2;+∞[ to [3;+∞[ that maps x to x^2
Exercise "f is increasing on [4;+∞["
 Conclusion: for all x ∈ [ 4 ; +∞ [,
               for all y ⩾ x,
                x⩾2 and y⩾2 and  f(y) ⩾ f(x)
Proof
 let x ∈  [ 4 ; +∞ [
 let y ⩾ x
 ◆ x ⩾ 2
 ◆ y ⩾ 2
 ◆ y^2 - x^2 = (y-x)*(y+x)
 ◆ y + x ⩾ 0
 ◆ y - x ⩾ 0
 ◆ (y + x)*(y - x) ⩾ 0
 ◆ y^2 - x^2       ⩾ 0
 ◆ x^2 ⩽ y^2
 ◆ f(x) ⩽ f(y)
□
```

Problematics:

- Proof assistants based on type theory manipulate type-total functions

$$f : \alpha \to \beta$$

- How to encode type-partial functions ?

We expect :

(i) f contains its domain ($[2; +\infty[$) and codomain ($[3; +\infty[$)

(ii) writing $f(x)$ is allowed because $x \in$ domain is provable

A first representation[3] guarantees (i) and (ii):

$$f : \uparrow D \to \uparrow F$$

where $\uparrow D := \sum_{x:\alpha} (x \in D)$ denotes the type of dependent pairs $\langle$`x:α,hx: x ∈ D`$\rangle$.

Syntactic sugar[4] :

- `f(3)` denotes `(f ⟨3,(by norm_num:(2:Real)⩽3)⟩).val`
- `let's define the function f from [2;+∞[ to [3;+∞[ that maps x to x^2` denotes:

```
def f:↑[2;+∞[→↑[3;+∞[:=fun ⟨x,(hx:x⩾2)⟩=>⟨x^2,((by nlinarith):(x^2 ⩾ 3))⟩
```

---

[3] described in C. Paulin's lecture `https://www.lri.fr/~paulin/LASER/coq-slides4.pdf`

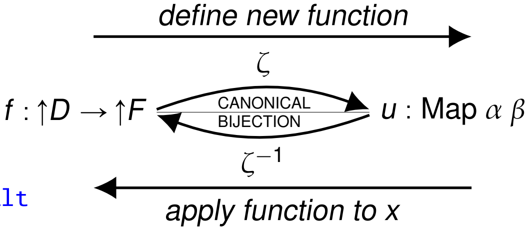
[4] Without sugar, discarded by Coen and Zoli 2007 in educational context

Drawbacks of first representation :

- A type appears to the user ($\uparrow D \to \uparrow F$)
- $f : \uparrow D \to \uparrow F$ and its derivative $f' : \uparrow D'_f \to \uparrow F'$ have distinct types, thus $\left\{ f \mid D'_f = D \text{ and } f = f' \right\}$ does not typecheck

Second representation:

```
structure Map (α β:Type) where
  func: α→β
  domain: Set α
  codomain: Set β
  prop: ∀x∈domain,func x∈codomain
  prop_out:∀x:α,x∉domain→func x=default
```



```
Theorem nn1even "7. Using (... or ...) : proof by cases"
    Assumptions: (for all integer n, n is even or n is odd)
    Conclusion: for all integer n, n*(n+1) is even
Proof
    let n be an integer
    ◆ n is even or n is odd
    ◆ if n is even then n*(n+1) is even
      proof
        assume n is even
        ◆ n*(n+1) is even by product_even
      □
    ◆ if n is odd then n*(n+1) is even
      proof
        assume n is odd
        ◆ n+1 is even
        ◆ n*(n+1) is even by product_even
      □
    ◆ n*(n+1) is even
□
```

Automation : example 1/3

```
Theorem nn1even "7. Using (... or ...) : proof by cases"
    Assumptions: (for all integer n, n is even or n is odd)
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Proof
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    ◆ n is even or n is odd
    ◆ if n is even then n*(n+1) is even
      proof
        assume n is even
        ◆ n*(n+1) is even by product_even
      □
    ◆ if n is odd then n*(n+1) is even
      proof
        assume n is odd
        ◆ n+1 is even
        ◆ n*(n+1) is even by product_even
      □
    ◆ n*(n+1) is even
□
```

Finds relevant assumption; eliminate ∀ in it

```
Theorem nn1even "7. Using (... or ...) : proof by cases"
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      □
    ◆ if n is odd then n*(n+1) is even
      proof
        assume n is odd
        ◆ n+1 is even
        ◆ n*(n+1) is even by product_even
      □
    ◆ n*(n+1) is even
□
```

Finds relevant assumption;
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As
assumptions in context

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Theorem nn1even "7. Using (... or ...) : proof by cases"
    Assumptions: (for all integer n, n is even or n is odd)
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      □
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□
```

Finds relevant assumption; eliminate ∀ in it

Ar

assumptions in context

The lemma give $(n+1)n$ even. Unifie $n(n+1)$ by commutativity.

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Theorem nn1even "7. Using (... or ...) : proof by cases"
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    ◆ if n is odd then n*(n+1) is even
      proof
        assume n is odd
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        ◆ n*(n+1) is even by product_even
      □
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□
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Finds relevant assumption; eliminate ∀ in it

As

assumptions in context

The lemma give $(n+1)n$ even. Unifie $n(n+1)$ by commutativity.

Eliminate

Automation : example 2/3

```
Theorem sqrt_2_is_irrational "√2 ∉ ℚ"
 Conclusion: √2 is not rational
Proof
 assume that √2 is rational
 ◆ there exists an integer p such that
    there exists a natural number q
     such that q≠ 0 and √2 = p/q and
     the statement (p is even and q is
    even) is false

 obtain such p
 obtain such q

 ◎ p^2 = (p/q)^2 * q^2
   _   = (√2)^2  * q^2
   _   =  2      * q^2
```

```
 ◆ p^2 is even
 ◆ p is even by n2_even_implies_n_even
 obtain an integer k such that p = 2*k
 ◎ q^2 = (2*q^2)/2
   _    = p^2 / 2
   _    = (2*k)^2 /2
   _    = 2 * k^2
 ◆ q^2 is even
 ◆ q is even by n2_even_implies_n_even
 ◆ Absurd
□
```

```
Theorem sqrt_2_is_irrational "√2 ∉ ℚ"
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 obtain such p
 obtain such q

 ◎ p^2 = (p/q)^2 * q^2
   _   = (√2)^2  * q^2
   _   =  2      * q^2
```

$q \neq 0$ *and* $\sqrt{2} = p/q$ *and* ...
broken into 3 fact

```
◆ p^2 is even
◆ p is even by n2_even_implies_n_even
obtain an integer k such that p = 2*k
◎ q^2 = (2*q^2)/2
  _   = p^2 / 2
  _   = (2*k)^2 /2
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◆ q^2 is even
◆ q is even by n2_even_implies_n_even
◆ Absurd
□
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 obtain such p
 obtain such q

 ◎ p^2 = (p/q)^2 * q^2
   _   = (√2)^2  * q^2
   _   =  2      * q^2
```

q ≠ 0 and √2 = p/q and . . .
broken into 3 fact

rewrite
compute

```
◆ p^2 is even
◆ p is even by n2_even_implies_n_even
obtain an integer k such that p = 2*k
◎ q^2 = (2*q^2)/2
  _     = p^2 / 2
  _     = (2*k)^2 /2
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◆ q is even by n2_even_implies_n_even
◆ Absurd
□
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 assume that √2 is rational
 ◆ there exists an integer p such that
    there exists a natural number q
     such that q≠ 0 and √2 = p/q and
     the statement (p is even and q is
    even) is false

 obtain such p
 obtain such q

 ◎ p^2 = (p/q)^2 * q^2
   _   = (√2)^2  * q^2
   _   =  2      * q^2
```

q ≠ 0 and √2 = p/q and...
broken into 3 fact

rewrite
compute

```
◆ p^2 is even
◆ p is even by n2_even_implies_n_even
obtain an integer k such that p = 2*k
◎ q^2 = (2*q^2)/2
  _    = p^2 / 2
  _    = (2*k)^2 /2
  _    = 2 * k^2
◆ q^2 is even
◆ q is even by n2_even_implies_n_even
◆ Absurd
□
```

Introduce ∃

```
Theorem sqrt_2_is_irrational "√2 ∉ ℚ"
 Conclusion: √2 is not rational
Proof
 assume that √2 is rational
 ◆ there exists an integer p such that
    there exists a natural number q
     such that q≠ 0 and √2 = p/q and
     the statement (p is even and q is
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 obtain such p
 obtain such q

 ◎ p^2 = (p/q)^2 * q^2
   _   = (√2)^2  * q^2
   _   =  2      * q^2
```

$q \neq 0$ and $\sqrt{2} = p/q$ and ...
broken into 3 fact

rewrite
compute

```
◆ p^2 is even
◆ p is even by n2_even_implies_n_even
obtain an integer k such that p = 2*k
◎ q^2 = (2*q^2)/2
  _   = p^2 / 2
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◆ q^2 is even
◆ q is even by n2_even_implies_n_even
◆ Absurd
□
```

Introduce ∃

Auto infers
$q \in \mathbb{Z}$

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◆ Absurd
□
```

Introduce ∃

Auto infers
$q \in \mathbb{Z}$

Introduce
Eliminate ⇒' in
relevant assumption

```
let's define the function f
 from [2;+∞[ to [3;+∞[ that maps x to x^2
Exercise "f is increasing on [4;+∞["
 Conclusion: for all x ∈ [ 4 ; +∞ [,
               for all y ⩾ x,
                x⩾2 and y⩾2 and  f(y) ⩾ f(x)
Proof
 let x ∈  [ 4 ; +∞ [
 let y ⩾ x
 ◆ x ⩾ 2
 ◆ y ⩾ 2
 ◆ y^2 - x^2 = (y-x)*(y+x)
 ◆ y + x ⩾ 0
 ◆ y - x ⩾ 0
 ◆ (y + x)*(y - x) ⩾ 0
 ◆ y^2 - x^2       ⩾ 0
 ◆ x^2 ⩽ y^2
 ◆ f(x) ⩽ f(y)
□
```

```
let's define the function f
 from [2;+∞[ to [3;+∞[ that maps x to x^2
```

Prove $\forall x \in [2;+\infty[, x^2 \in [3;+\infty[$

```
Exercise "f is increasing on [4;+∞["
 Conclusion: for all x ∈ [ 4 ; +∞ [,
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 ◆ (y + x)*(y - x) ⩾ 0
 ◆ y^2 - x^2        ⩾ 0
 ◆ x^2 ⩽ y^2
 ◆ f(x) ⩽ f(y)
□
```

Prove $\forall x \in [2;+\infty[, x^2 \in [3;+\infty[$

Prove $x \in [2;+\infty[$

Prove $y \in [2;+\infty[$

Automation : example 3/3

```
let's define the function f
 from [2;+∞[ to [3;+∞[ that maps x to x^2
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□
```

Prove $\forall x \in [2; +\infty[, x^2 \in [3; +\infty[$

Prove $x \in [2; +\infty[$

Prove $y \in [2; +\infty[$

Prove $y \in [2; +\infty[$

Prove $x \in [2; +\infty[$

```
Theorem nn1even' "7. Using (... or ...) : proof by cases"
    Assumptions: (for all integer n, n is even or n is odd)
    Conclusion: for all integer n, n*(n+1) is even
Proof
  let n ∈ ℤ
  ◆ n is even or n is odd
  ?
■
```

1 goal

```
assumption1 : ∀ {n : Number}, if (n is integer
n : Number
assumption2 : n is integer
assumption3 : n is even or n is odd
⊢ n * (n + 1) is even
```

1

Use n is even or n is odd

▼ Messages (1)

▼ 01_HighSchoolProgression.lean:560:2

"Proof unfinished."

Widgets

```
Theorem nn1even' "7. Using (... or ...) : proof by cases"
    Assumptions: (for all integer n, n is even or n is odd)
    Conclusion: for all integer n, n*(n+1) is even
Proof
  let n ∈ ℤ
  ◆ n is even or n is odd
  ◆ if n is even then n * (n + 1) is even
    proof
      assume n is even
      ?
    ■
  ◆ if n is odd then n * (n + 1) is even
    proof
      assume n is odd
      ?
    ■
  ◆ n * (n + 1) is even by cases
■
```

```
1 goal
assumption1 : ∀ {n : Number}, if (n is integer
n : Number
assumption2 : n is integer
assumption3 : n is even or n is odd
⊢ n * (n + 1) is even
```

1

Use n is even or n is odd

▼ Messages (1)

▼ 01_HighSchoolProgression.lean:560:2

Conclusion

Yalep, geared towards high-school students, features:

- A minimal language declarative language based on forward chaining
- Enough automation to allow most basic proof steps left unjustified
- A mechanism to hide type foundations of the underlying P.A., namely:
  - Transparent handling of user number sets $\mathbb{N}$,$\mathbb{Z}$,$\mathbb{Q}$,$\mathbb{R}$.
  - Transparent manipulation of type-partial functions

Try it!
Web interface:

Git:

Future work

- Automation : rationalize, optimize, limit, configure
- Feedback (pretty printer, error messages)
- Cover high school curriculum (including Première, Terminale)

Demo

Questions?

Thank you !

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