# On Propositional Model Counting and Enumeration and Automated Model Building in Predicate Logic

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EuroProofNet Workshop on Program Verification

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# Where Do These Things Come Into Play?

Model Checking

Probabilistic Reasoning

Planning

Model Enumeration

Model Counting

Weighted Model Integration

Satisfiability Modulo Theories

Cryptography

Hardware Verification

Automated Theorem Proving

Automated Model Building

Software Verification

Model Checking

Circuit Design

Preimage Computation

Bayesian Networks

Model Theory

Predicate Abstraction

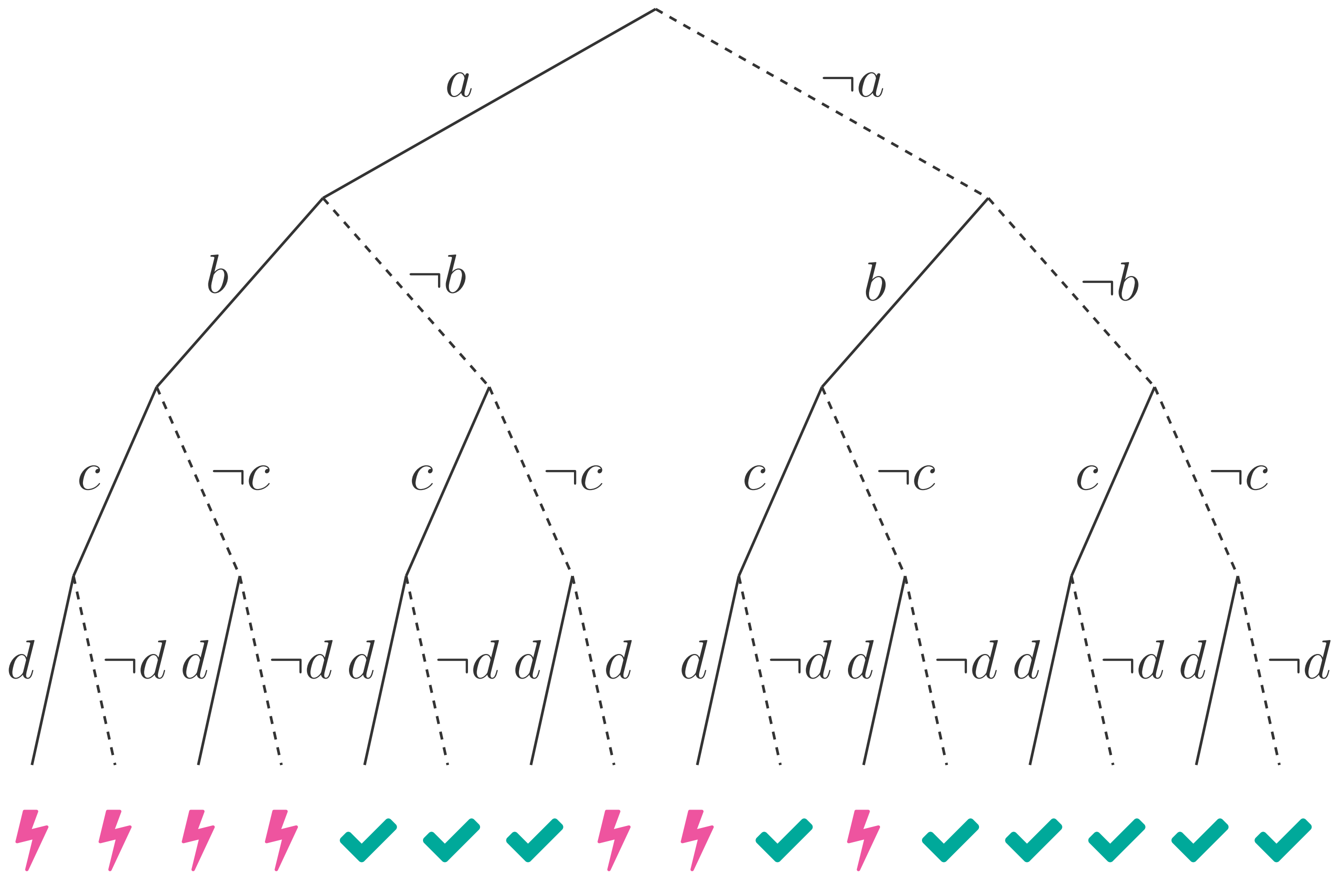
Product Configuration

# Counting and Enumerating Propositional Models

# Challenge: Search Space Need be Explored Exhaustively

| $n$ | $2^{\|n\|}$ assignments |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 10 | $1'024$ |
| 100 | $\approx 1.27 \cdot 10^{30}$ |
| 1000 | $\approx 1.07 \cdot 10^{301}$ |

$$F = (\neg a \lor \neg b) \land (a \lor \neg b \lor \neg d) \land (\neg a \lor b \lor c \lor d)$$



# State-Of-The-Art Solution for Model Counting: Component-Based Reasoning

$$F = \underbrace{(a \vee b)}_{\mathcal{C}_1} \wedge \underbrace{(c \vee d)}_{\mathcal{C}_2}$$

$$\Downarrow$$

$$\#F = 9 = \#\mathcal{C}_1 \cdot \#\mathcal{C}_2$$


R.J. Bayardo, J.D. Pehoushek. *Counting Models Using Connected Components".* AAAI'00.
T. Sang et al. *Combining Component Caching and Clause Learning for Effective Model Counting.* SAT'04.
M. Thurley. *sharpSAT — Counting Models with Advanced Component Caching and Implicit BCP.* SAT'06.
J. Burchard, T. Schubert, B. Becker. *Laissez-Faire Caching for Parallel #SAT Solving.* SAT'15.
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👍 reduces work in individual computation

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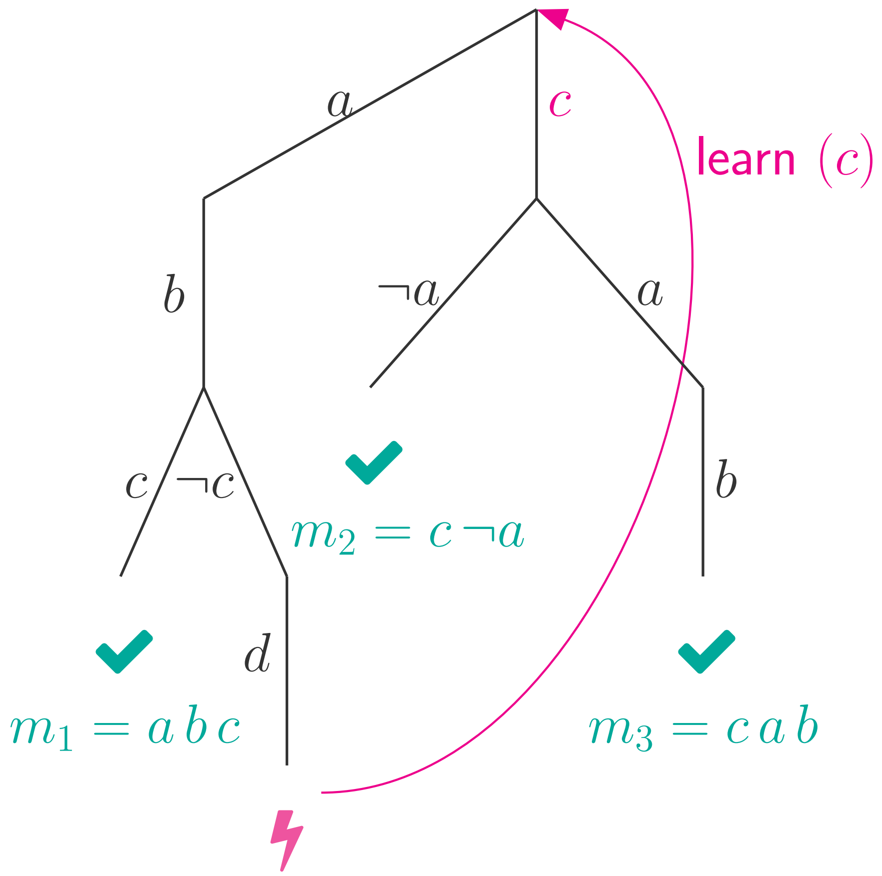
# Challenge: Avoid Finding Models Multiple Times in CDCL With Backjumping

$$F = (\neg a \vee b) \wedge (c \vee d) \wedge (c \vee \neg d) \wedge (c)$$





J.P. Marques-Silva, K.A. Sakallah. *GRASP—A New Search Algorithm for Satisfiability.* ICCAD'96.
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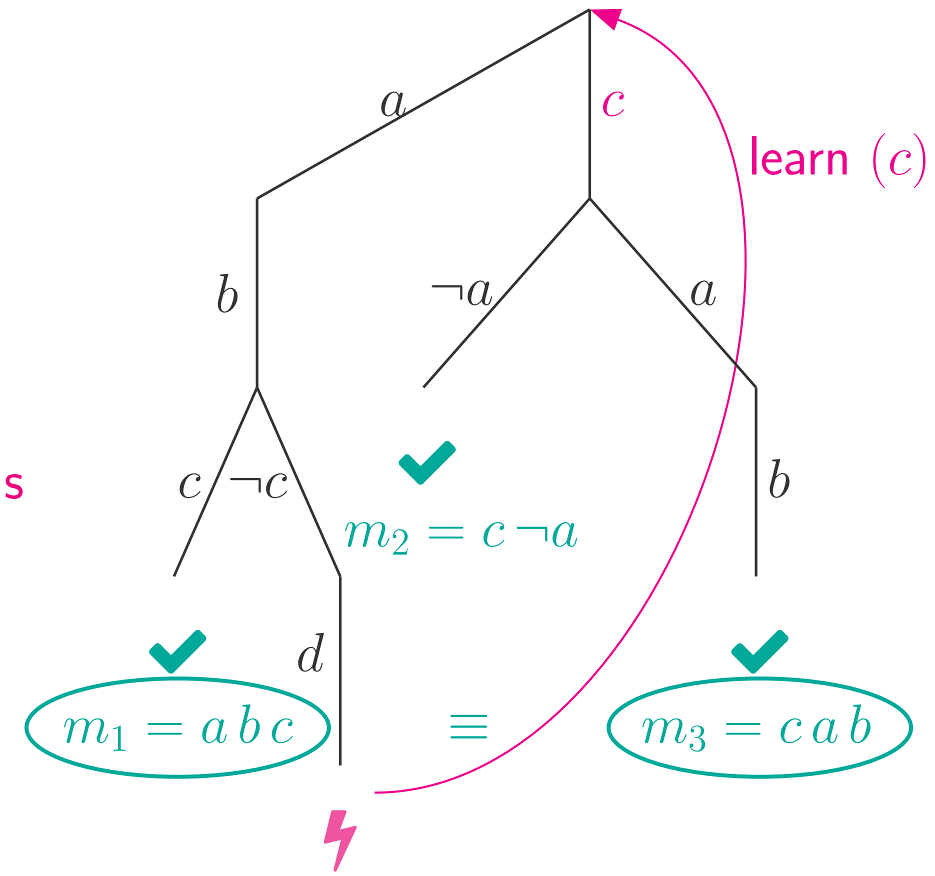
M.W. Moskewicz, C.F. Madigan, Y. Zhao, L. Zhang, S. Malik. *Chaff: Engineering an Efficient SAT Solver.* DAC'01.

# Challenge: Avoid Finding Models Multiple Times in CDCL With Backjumping

$$F = (\neg a \vee b) \wedge (c \vee d) \wedge (c \vee \neg d) \wedge (c)$$

- learn from conflicts
- might find redundant models





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# State-Of-The-Art Solution: Blocking Clauses and DPLL-Style Backtracking

## Blocking clauses

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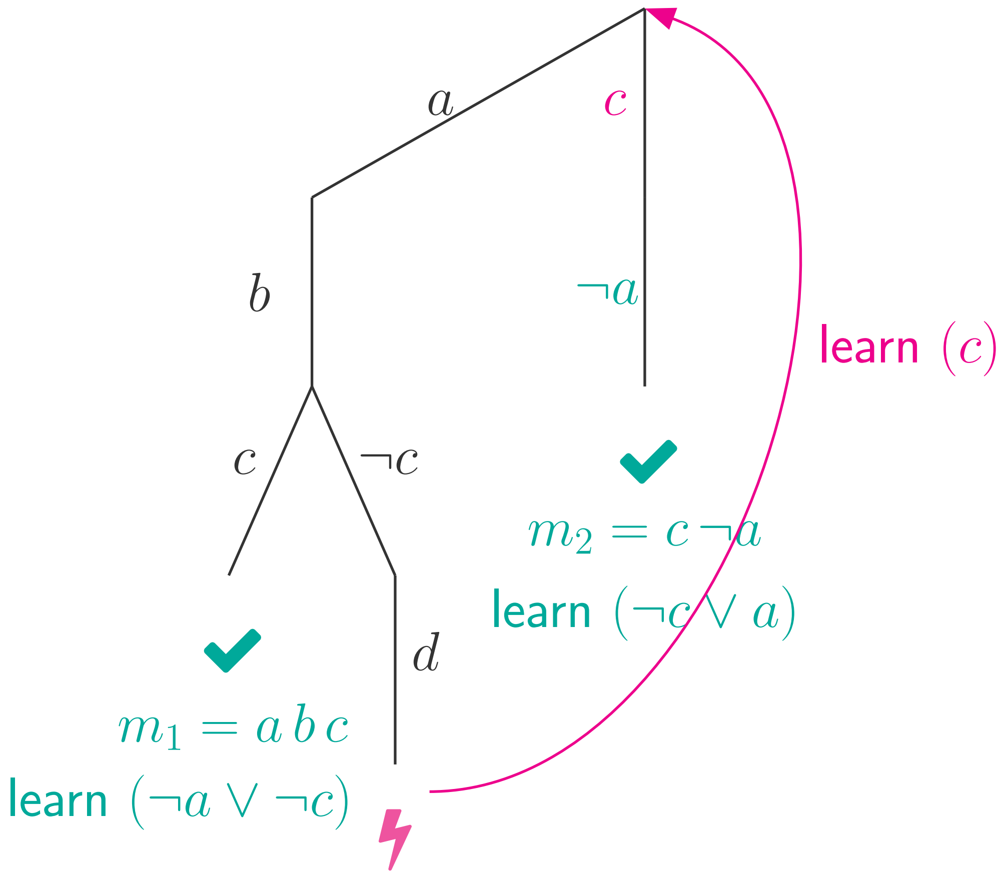



K.L. McMillan. *Applying SAT Methods in Unbounded Symbolic Model Checking.* CAV'02.
A.J. dos Reis Morgado, J. Marques-Silva. *Good Learning and Implicit Model Enumeration.* ICTAI'05.
S. Möhle, A. Biere. *Dualizing Projected Model Counting.* ICTAI'18.
S. Möhle, R. Sebastiani, A. Biere. *Four Flavors of Entailment.* SAT'20.

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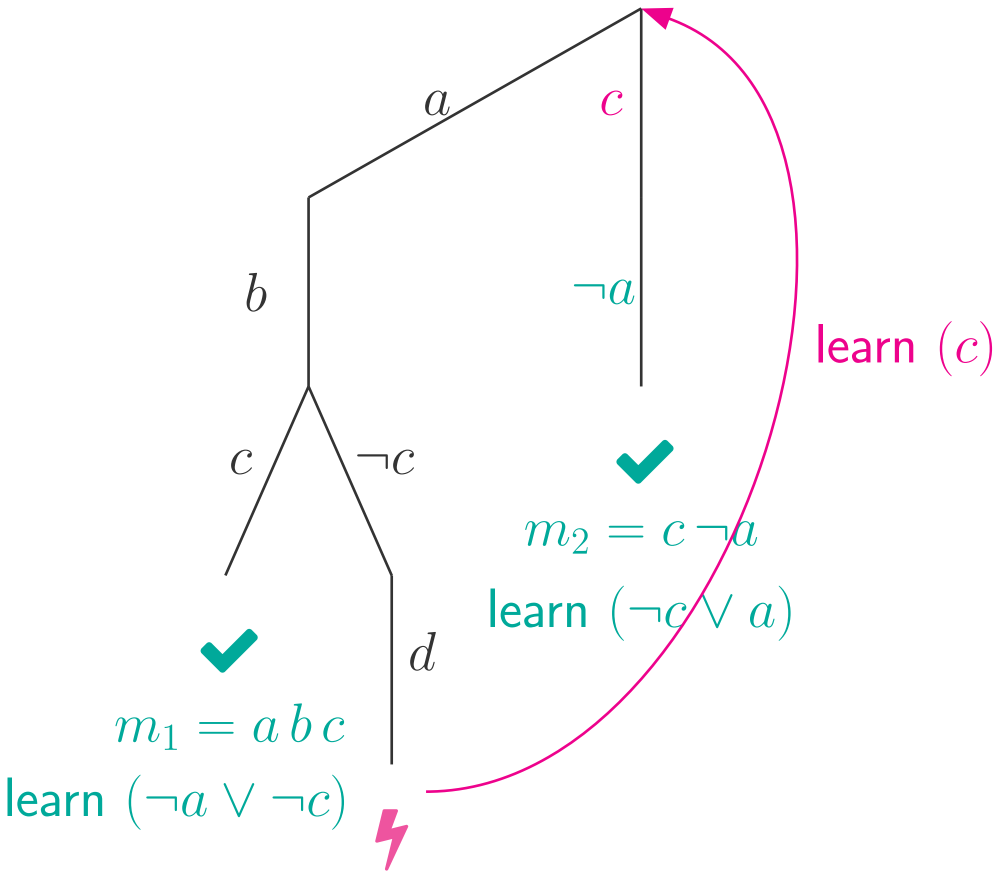


- finds no redundant models
- learns from conflicts
- blocking clauses must be kept $\Rightarrow$ formula blowup

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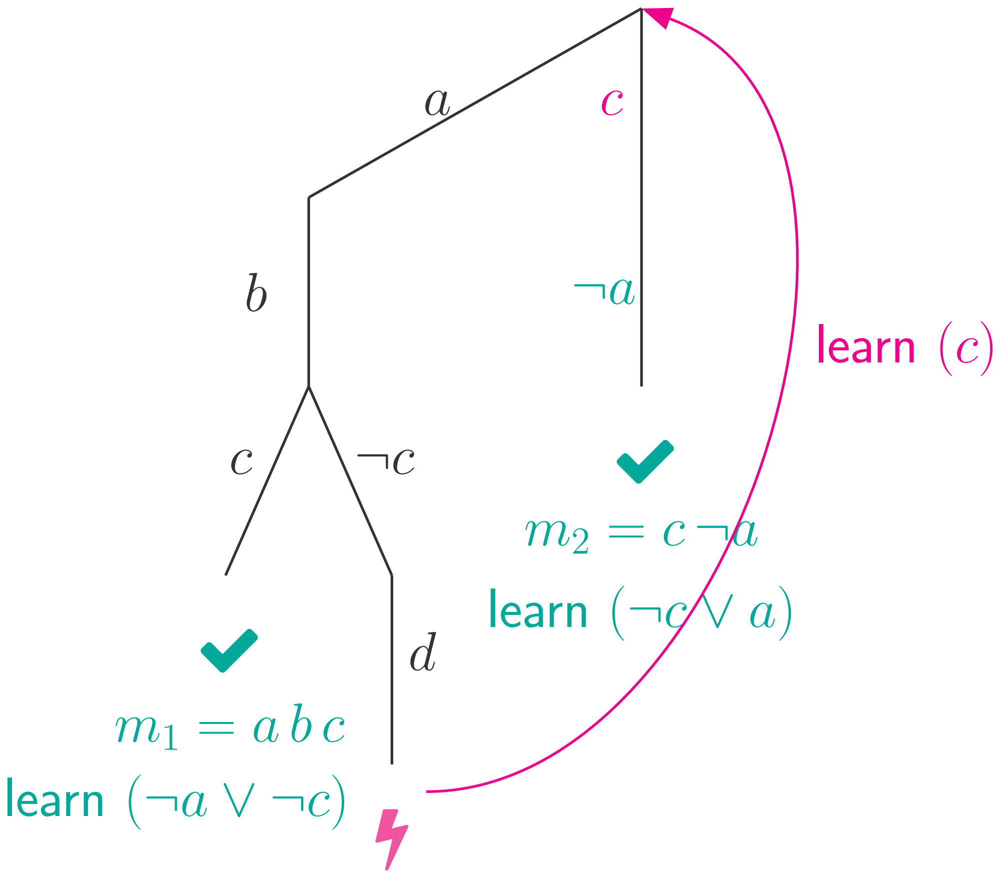
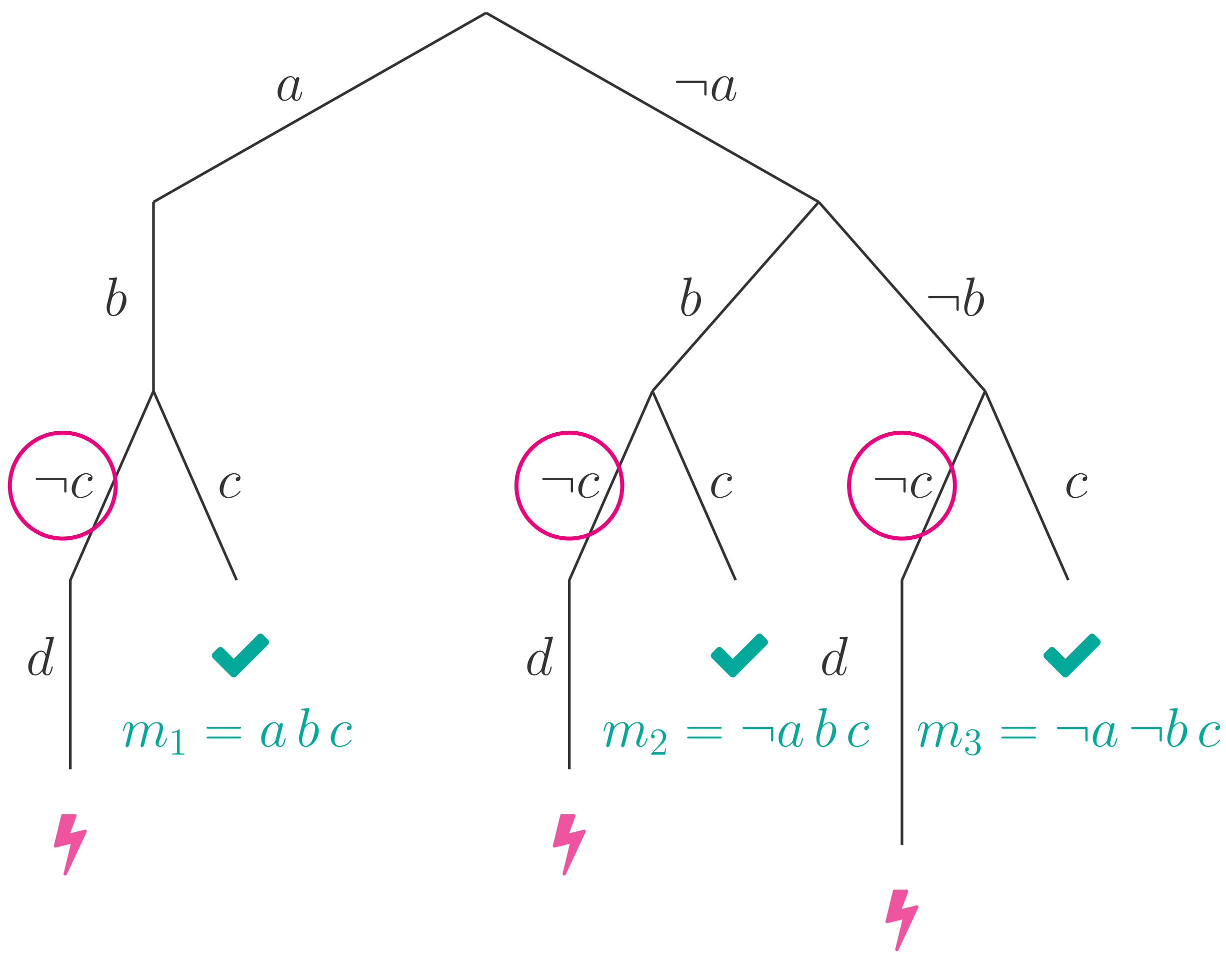



Chronological backtracking

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E. Birnbaum, E.L. Lozinskii. *The Good Old Davis-Putnam Procedure Helps Counting Models.* JAIR, 1999.

# State-Of-The-Art Solution: Blocking Clauses and DPLL-Style Backtracking

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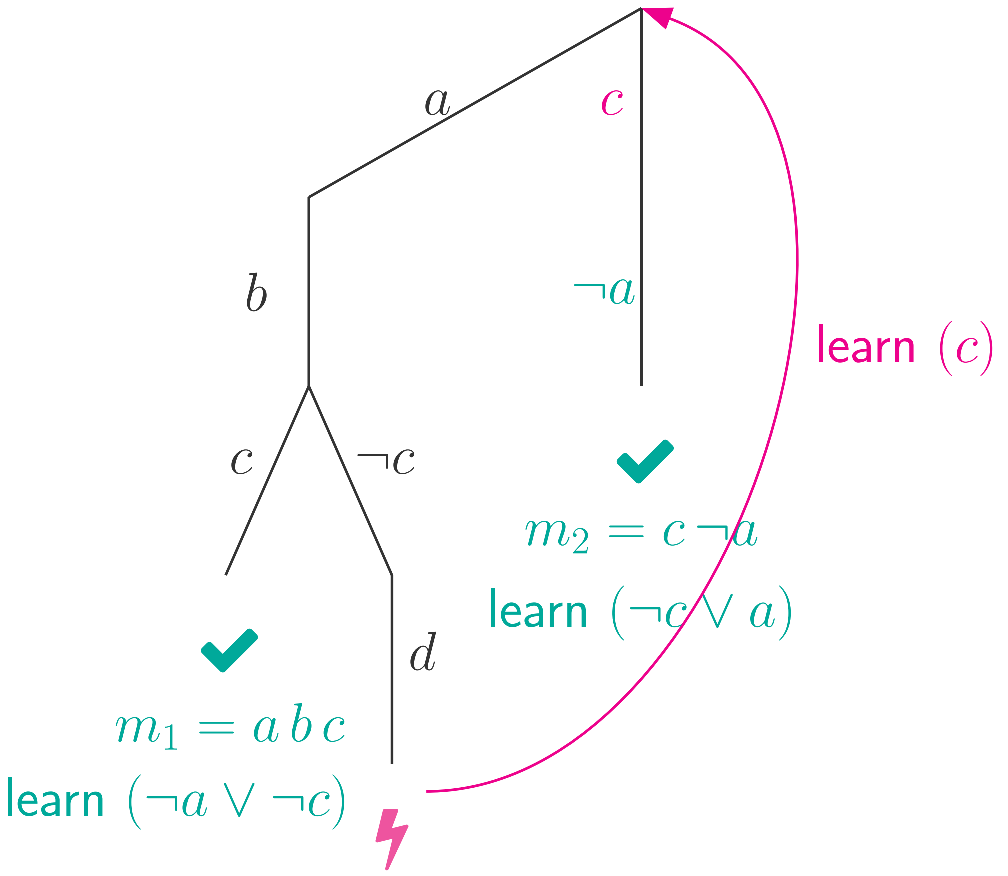


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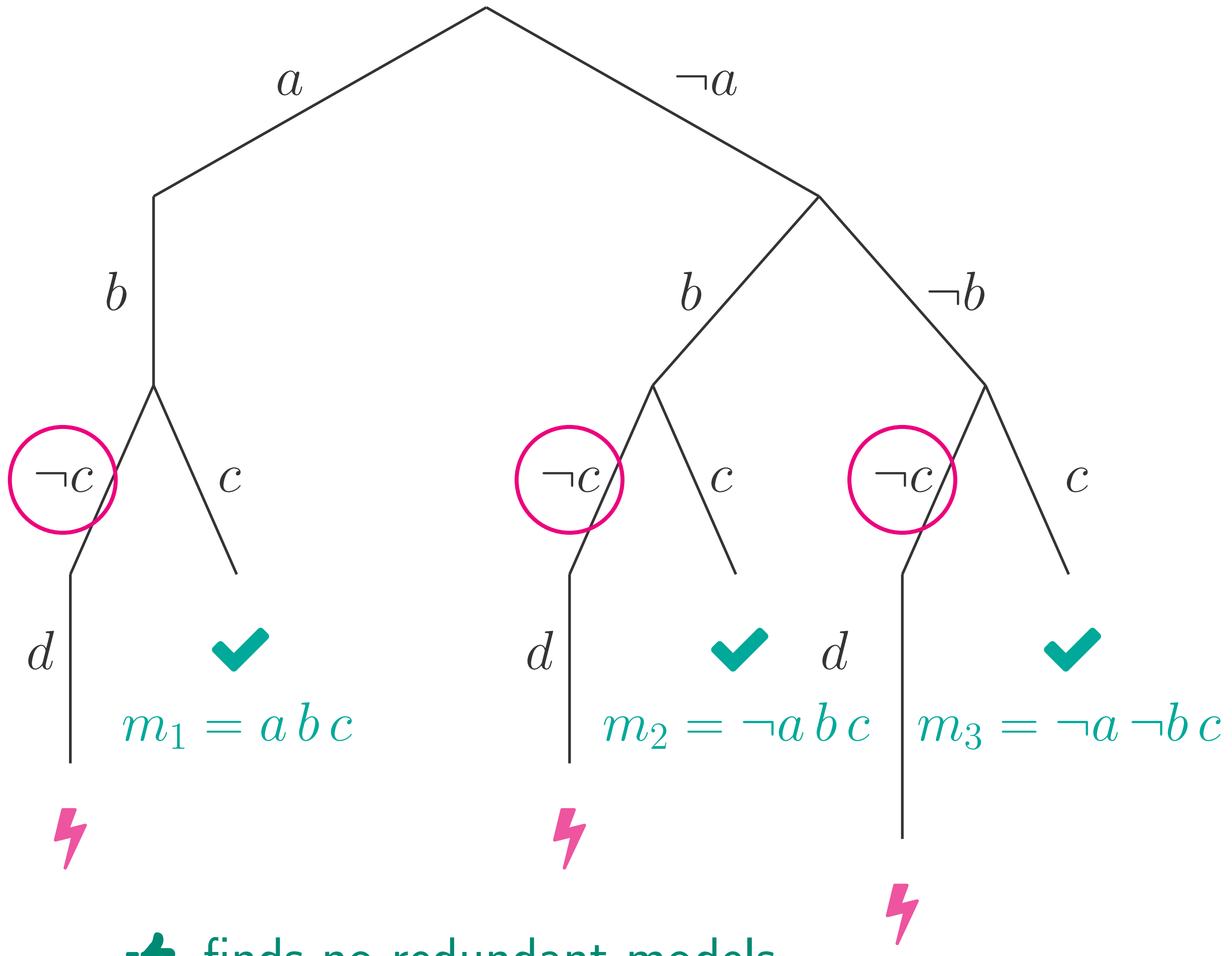


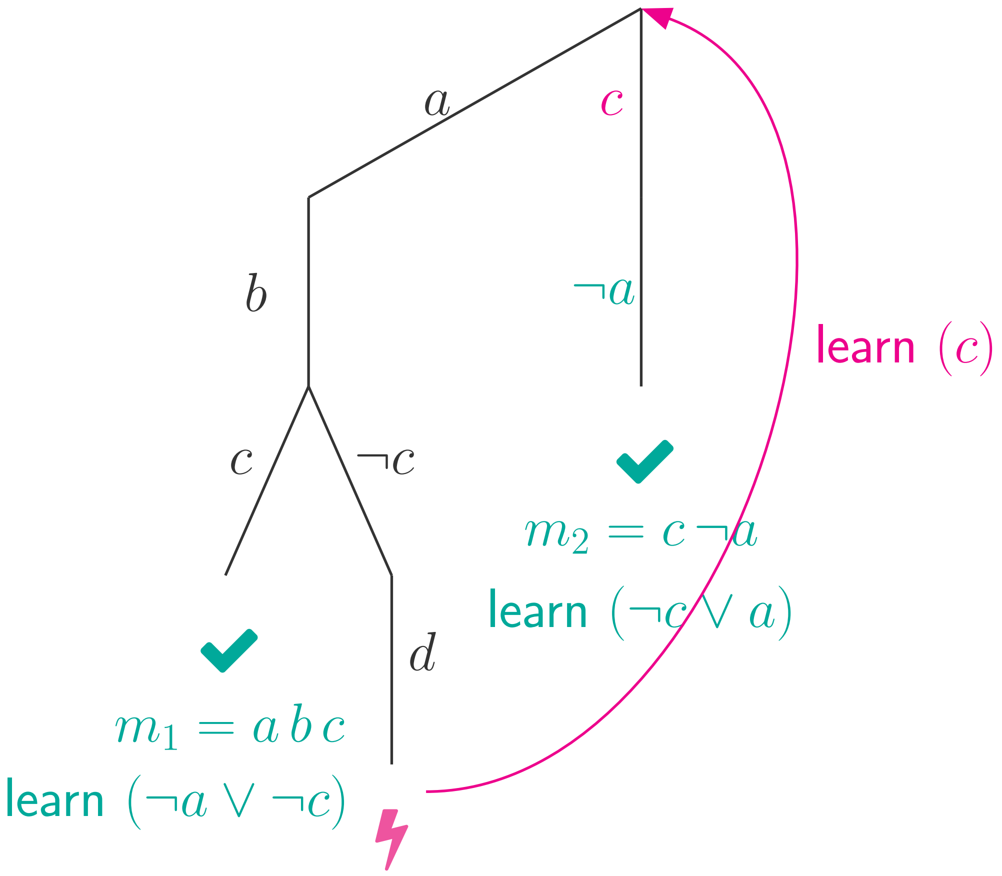

- finds no redundant models
- no formula blowup
- no learning from conflicts

# State-Of-The-Art Solution: Blocking Clauses and DPLL-Style Backtracking

$$F = (\neg a \vee b) \wedge (c \vee d) \wedge (c \vee \neg d)$$

Blocking clauses

Chronological CDCL

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A. Nadel, V. Ryvchin, *Chronological Backtracking.* SAT'18.
S. Möhle, A. Biere, *Backing Backtracking.* SAT'19.
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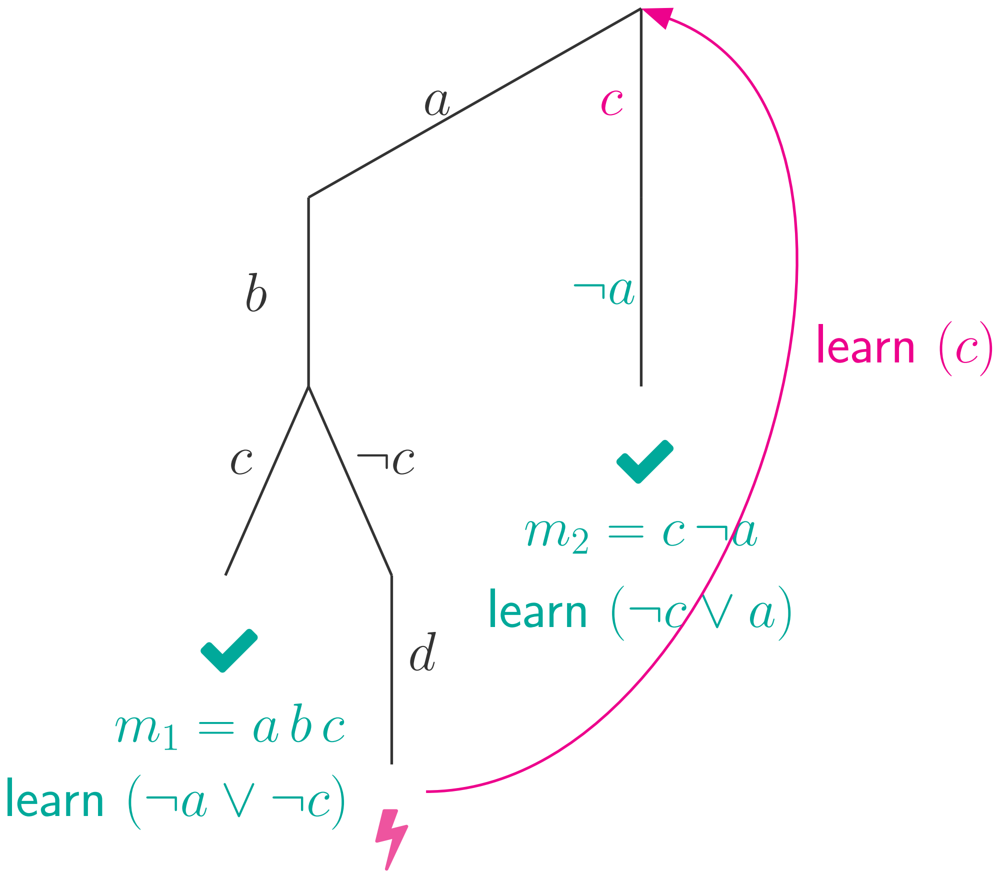


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$F = (\bar{p} \vee q) \wedge (p \vee q)$

var$(F) = \{p, q, r\}$

Rules

$M = (r \wedge q) \vee (\bar{r} \wedge p \wedge q) \vee (\bar{r} \wedge \bar{p} \wedge q) = C_1 \vee C_2 \vee C_3$

$M \equiv F$ and $\#M = \sum_{i=1}^{3} 2^{|V - C_i|} = 4 = \#F$

Generalizing, $\boxed{\#F = \sum_{C \in M} 2^{|V-C|} \text{ and } F \equiv \bigvee_{C \in M} C}$ and

$M$ is a Disjoint-Sum-of-Products (DSOP) representation of $F$

- $M$ is a disjunction of conjunctions of literals (cubes)
- The cubes in $M$ are pairwise contradicting
- $M$ is logically equivalent to $F$
- $M$ is not unique

S. Möhle, A. Biere. *Combining Conflict-Driven Clause Learning and Chronological Backtracking for Propositional Model Counting.* GCAI'19.

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Generalizing, $\boxed{\#F = \sum_{C \in M} 2^{|V-C|} \text{ and } F \equiv \bigvee_{C \in M} C}$ and

$M$ is a Disjoint-Sum-of-Products (DSOP) representation of $F$ $\longleftarrow$ we did some kind of knowledge compilation, right?

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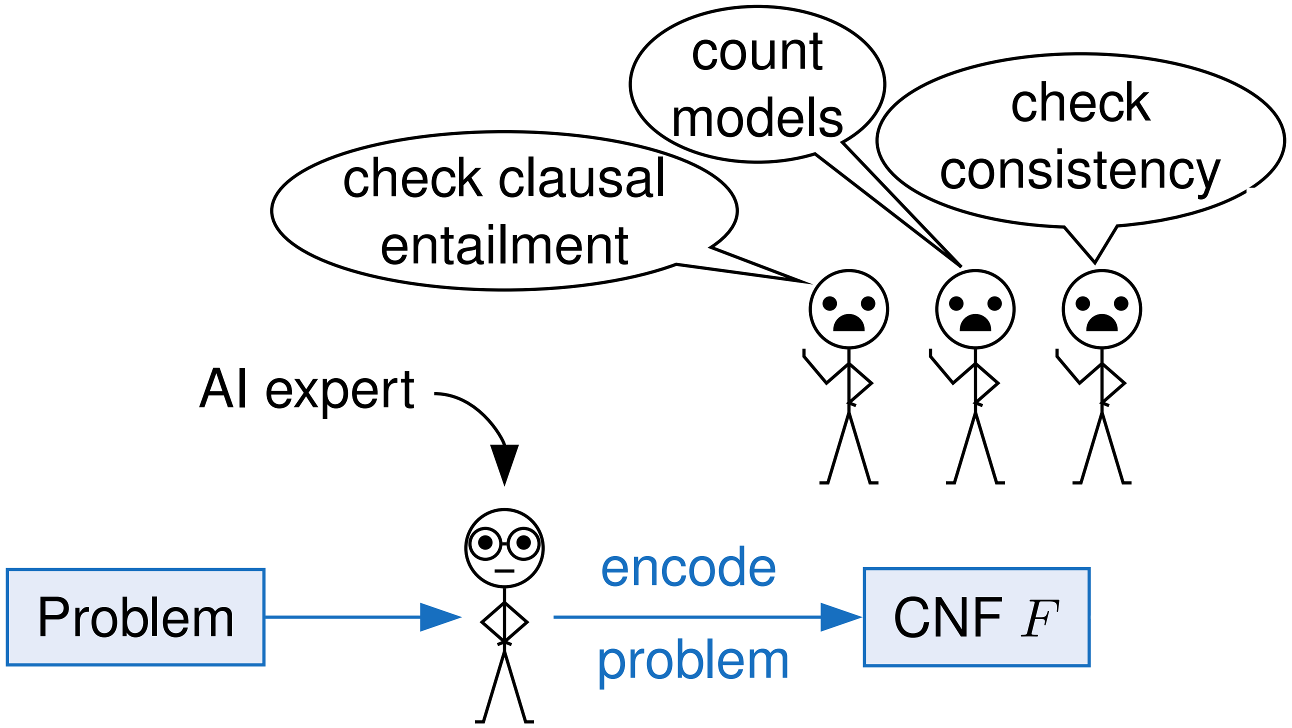
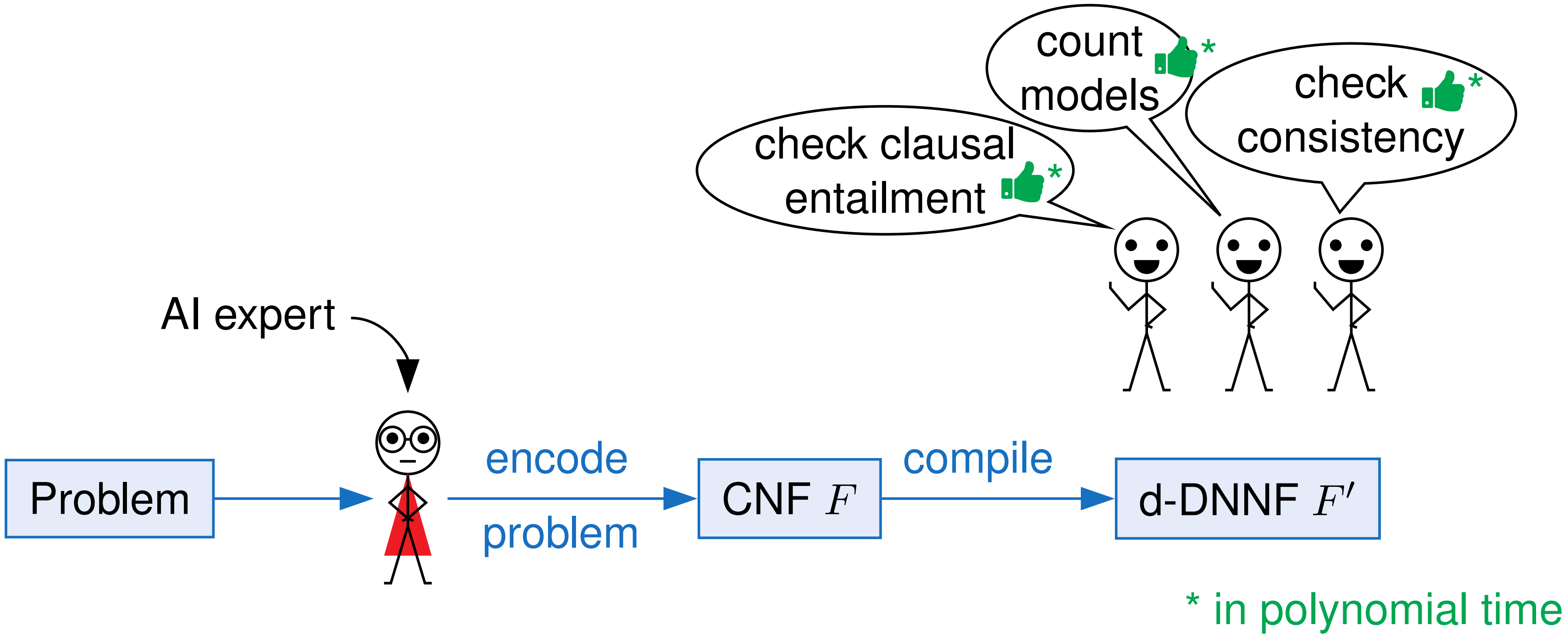
# Why Is Knowledge Compilation Interesting, Again?

# What's the Problem with Conjunctive Normal Form (CNF)?



# What's the Problem with Conjunctive Normal Form (CNF)?



# Deterministic Decomposable Negation Normal Form (d-DNNF)

d-DNNF: $F = (a \wedge (b \vee (\neg b \wedge c)) \wedge d) \vee (\neg a \wedge b)$

- negations in front of variables
- for all conjunctions: conjuncts do not share variables
- for all disjunctions: disjuncts are pairwise contradicting

# Model Counting in d-DNNF

(1) $F = G \wedge H \implies \#F = \#G \cdot \#H$ provided $\mathsf{var}(G) \cup \mathsf{var}(H) = \mathsf{var}(F)$ and $\mathsf{var}(G) \cap \mathsf{var}(H) = \emptyset$

(2) $F = C \vee D \implies \#F = 2^{|\mathsf{var}(F)| - |\mathsf{var}(C)|} + 2^{|\mathsf{var}(F)| - |\mathsf{var}(D)|}$ provided $C \wedge D \equiv \bot$

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$$F = \underline{(a \wedge (b \vee (\neg b \wedge c)) \wedge d)} \vee \underline{(\neg a \wedge b)} \tag{2}$$

$$\#F = \#(\underline{a} \wedge (\underline{b \vee (\neg b \wedge c)}) \wedge \underline{d}) \cdot 2^0 + \#(\underline{\neg a} \wedge \underline{b}) \cdot 2^2 \tag{1}$$

$$= [\#(a) \cdot \#(\underline{b} \vee \underline{(\neg b \wedge c)}) \cdot \#(d)] \cdot 2^0 + [\#(\neg a) \cdot \#(b)] \cdot 2^2 \tag{2}$$

$$= [1 \cdot [\#(b) \cdot 2^1 + \#(\underline{\neg b} \wedge \underline{c})] \cdot 2^0] + [1 \cdot 1] \cdot 2^2 \tag{1}$$

$$= [1 \cdot [1 \cdot 2^1 + [\#(\neg b) \cdot \#(c)] \cdot 2^0] + 2^2$$

$$= [1 \cdot [1 \cdot 2^1 + [1 \cdot 1] \cdot 2^0] + 2^2 = 7$$

# CNF vs. d-DNNF — the Model Counting Case

CNF: $F = (\neg a \vee b \vee c) \wedge (\neg a \vee d) \wedge (a \vee b)$

d-DNNF: $F' = (a \wedge (b \vee (\neg b \wedge c)) \wedge d) \vee (\neg a \wedge b)$

# CNF vs. d-DNNF — the Model Counting Case

CNF: $F = (\neg a \lor b \lor c) \land (\neg a \lor d) \land (a \lor b)$

$\#F = ?$ not that easy

d-DNNF: $F' = (a \land (b \lor (\neg b \land c)) \land d) \lor (\neg a \land b)$

$\#F' = (1 \cdot (1 \cdot 2^1 + 1 \cdot 1 \cdot 2^0) \cdot 1) \cdot 2^0 + (1 \cdot 1) \cdot 2^2 = 7$

# The Task: Knowledge Compilation

$$F \longrightarrow \mathsf{KC}(F) \equiv F$$

# The Task: CNF-to-d-DNNF Compilation

$$F \longrightarrow \text{KC}(F) \equiv F$$

this work

conjunctive normal form (CNF)

deterministic decomposable negation normal form (d-DNNF)

# Automated Model Building

# What do Knowledge Compilation and Automated Model Building Have in Common?

Knowledge compilation:
Translate a logical formula into another language *in which some tasks of interest are executable efficiently.*

Automated model building:
Find and finitely represent typically infinite models of a formula *facilitating their computational processing.*

$$\Downarrow$$

The focus is on a convenient representation.


M. Cadoli, F. Donini. *A Survey on Knowledge Compilation.* AI commun., 1997.
A. Darwiche, P. Marquis. *A Knowledge Compilation Map.* J. Artif. Intell. Res., 2002.
R. Caferra, A. Leitsch, N. Peltier. *Automated Model Building.* Kluwer, 2004.

# Automated Model Building Postulates

Let $\Sigma$ be a vocabulary and $F$ be a formula over $\Sigma$. Then a model representation formalism should ideally meet the following postulates:

- Uniqueness. Each model representation $M$ specifies a single interpretation over $\Sigma$.
- Atom Test. There exists a fast procedure to evaluate arbitrary ground atoms over the signature $\Sigma$ in $M$.
- Formula Evaluation. There exists an algorithm deciding the truth value of an arbitrary formula over $\Sigma$ in $M$.
- Equivalence Test. There exists an algorithm deciding whether two representations $M$ and $M'$ over $\Sigma$ describe the same interpretation.

C.G. Fermüller, A. Leitsch. *Hyperresolution and automated model building.* J. Log. Comput. 6(2), 1996.
R. Caferra, A. Leitsch, N. Peltier. *Automated Model Building.* Kluwer, 2004.

# The Satisfiability Problem of First-Order Logic

| | | | | |
|---|---|---|---|---|
| Signature | $\Sigma = (\Omega, \Pi)$ | where | $\Omega$ | set of function symbols of arity $\geq 0$ |
| | | | $\Pi$ | set of predicate symbols of arity $\geq 0$ |
| | | | $\mathcal{X}$ | set of variables |

Term $t \in T(\Sigma, \mathcal{X})$ where $t$ is either a $x \in \mathcal{X}$ or $f(t_1, \dots, t_n)$ with $f \in \Omega$ and $t_1, \dots, t_n \in T(\Sigma, \mathcal{X})$

Ground term $t \in T(\Sigma)$ where $t$ contains no variables

Atom $P(t_1, \dots, t_n)$ where $P \in \Pi$ and $t_1, \dots, t_n \in T(\Sigma, \mathcal{X})$

Literal an atom or its negation

Clause a disjunction of literals

Substitution $\sigma : \mathcal{X} \to T(\Sigma, \mathcal{X})$ such that $\sigma(x) \neq x$ for only finitely many variables

Unification terms $s$ and $t$ are unifiable if there exists a substitution $\sigma$ such that $\sigma(s) = \sigma(t)$

# Solving the Satisfiability Problem of First-Order Logic

$\Sigma = (\{a/0, f/1\}, \{P/1\})$ $\qquad$ $N = \{\underbrace{P(a)}_{C_1}, \underbrace{\neg P(x) \lor P(f(x))}_{C_2}\}$

1. Propagate $P(a)$ from $C_1$: $\Gamma = P(a)$
2. Apply $\sigma = \{x \mapsto a\}$ to $C_2$ and propagate $P(f(a))$: $\Gamma = P(a)\, P(f(a))$
3. Apply $\sigma = \{x \mapsto f(a)\}$ to $C_2$ and propagate $P(f(f(a)))$: $\Gamma = P(a)\, P(f(a))\, P(f(f(a)))$
4. $\ldots$

But the infinite model of $N$ could be just represented by $P(x)$...

A. Fiori, C. Weidenbach. *SCL Clause Learning from Simple Models.* CADE'19.

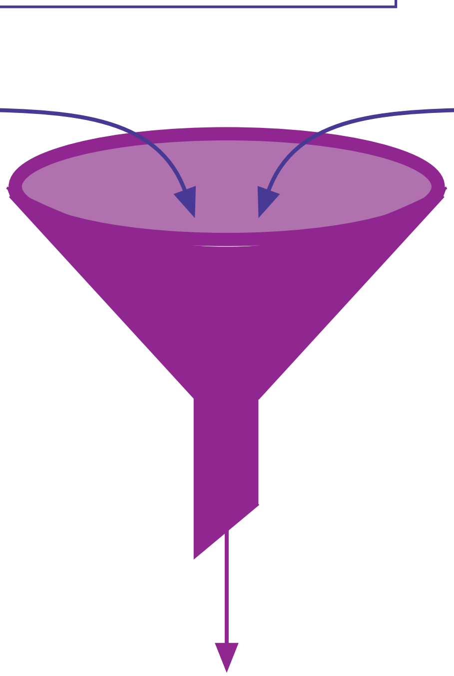
# Representing Interpretations as Set of Literals

$$N = \{P(a, f(x), a), P(x, f(x), x) \lor \neg P(x, g(x), x)\}$$

$$\Gamma = [P(a, f(a), a), \neg P(a, g(a), a)]$$

$$\Sigma = (\{a/0, f/1, g/1\}, \{P/3\})$$

$$\Gamma \models \mathsf{gnd}(N)^{\preceq \neg P(a,(f(a),a)}$$



Knuth-Bendix ordering with

$$\neg P \succ P \succ f \succ g \succ a$$

$$w(P) = w(f) = w(g) = w(a) = 1$$

$$M = \{P(x_1, a, x_3), P(x_1, f(y_1), x_3), \neg P(x_1, g(y_2), x_3)\} \models N$$

# Example

Given $N = \{P(a, f(x), a), P(x, f(x), x) \lor P(x, g(x), x)\}$ and a trail $\Gamma = P(a, f(a), a)\, \neg P(a, g(a), a)$

Wanted: representation of models of $N$ = generalization of $\Gamma$

| $\Gamma$ | $\Delta$ |
|---|---|
| $P(a, f(a), a)$<br>$\neg P(a, g(a), a)$ | |
| $\mathcal{I} \models \Gamma$ | $M \models \Delta$ |
| $P(x_1, x_2, x_3)$ | |

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M. Bromberger, F. Krasnopol, S. Möhle, Chr. Weidenbach. *First-Order Automatic Literal Model Generation.* IJCAR'24.

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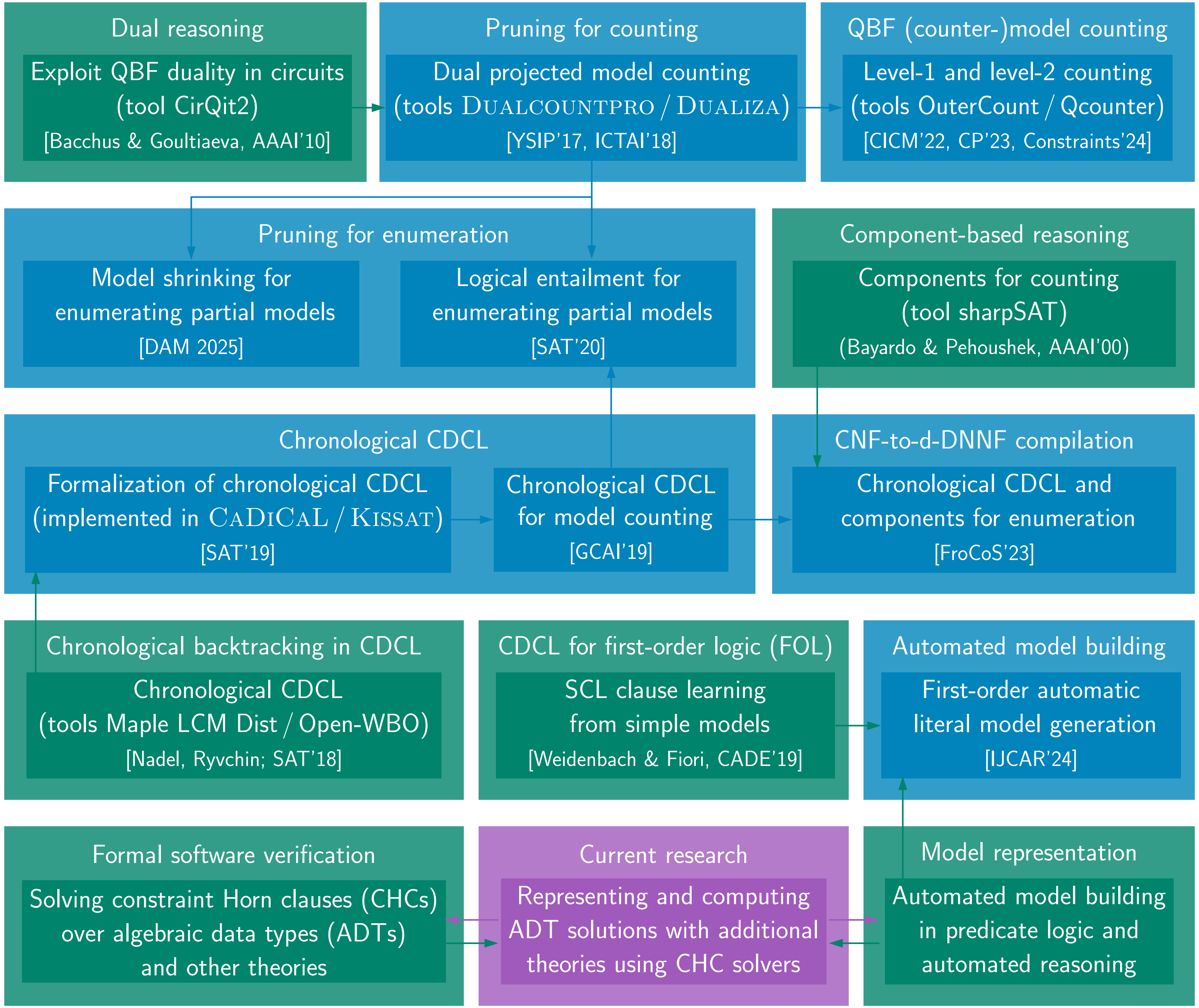
# Some Context



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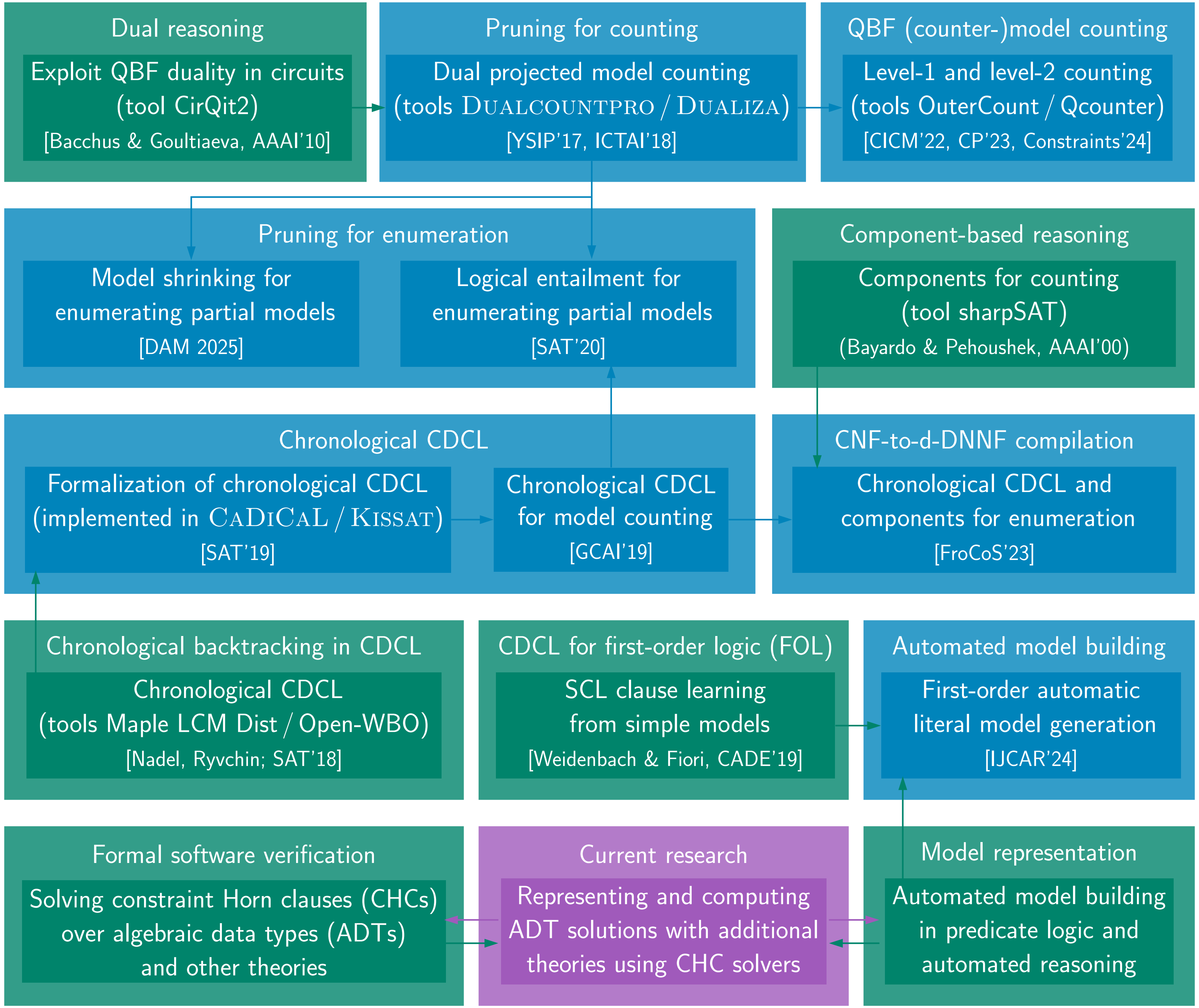


Thank you for your attention ☺

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# From DSOP to d-DNNF

$$F = (\neg a \vee b \vee c) \wedge (\neg a \vee d) \wedge (a \vee b)$$

$$\mathsf{DSOP}(F) = ({\color{cyan}a} \wedge b \wedge {\color{magenta}d}) \vee ({\color{cyan}a} \wedge \neg b \wedge c \wedge {\color{magenta}d}) \vee (\neg a \wedge b)$$

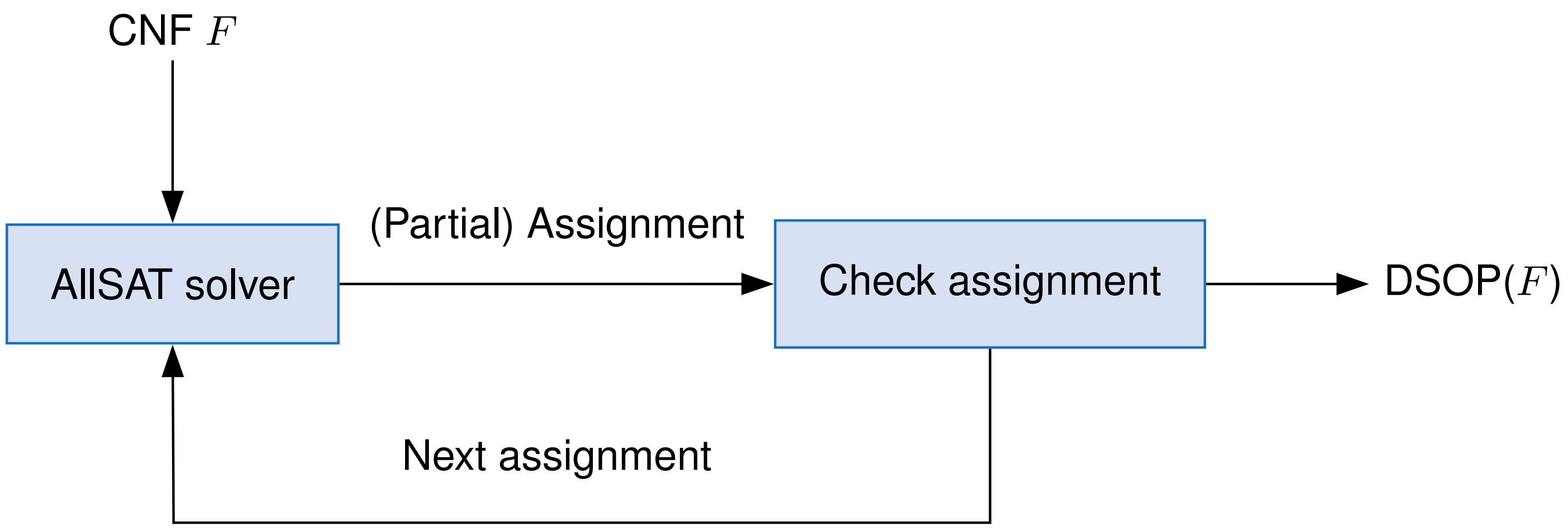
# From DSOP to d-DNNF

$$F = (\neg a \vee b \vee c) \wedge (\neg a \vee d) \wedge (a \vee b)$$

$$\text{DSOP}(F) = (a \wedge b \wedge d) \vee (a \wedge \neg b \wedge c \wedge d) \vee (\neg a \wedge b)$$

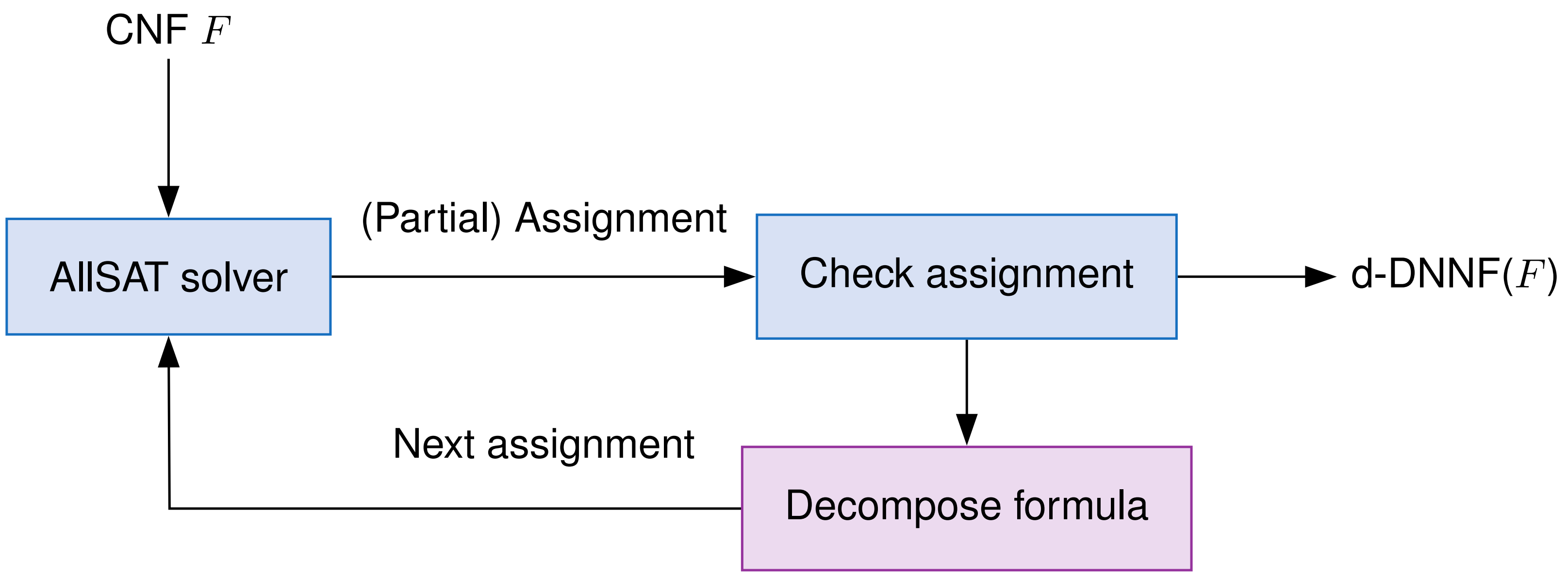
$$\text{d-DNNF}(F) = (a \wedge (b \vee (\neg b \wedge c)) \wedge d) \vee (\neg a \wedge b)$$

# CNF-to-d-DNNF Compilation: Main Idea




DSOP = disjoint sum-of-products

# CNF-to-d-DNNF Compilation: Main Idea



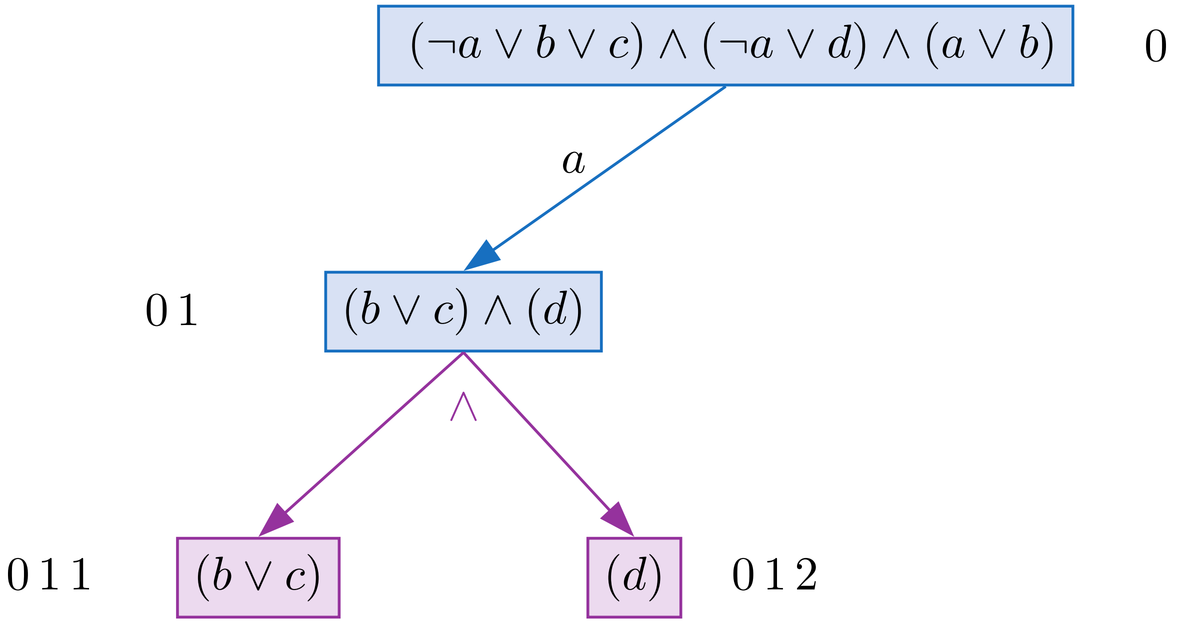
# Example

$$\boxed{(\neg a \vee b \vee c) \wedge (\neg a \vee d) \wedge (a \vee b)} \qquad 0$$

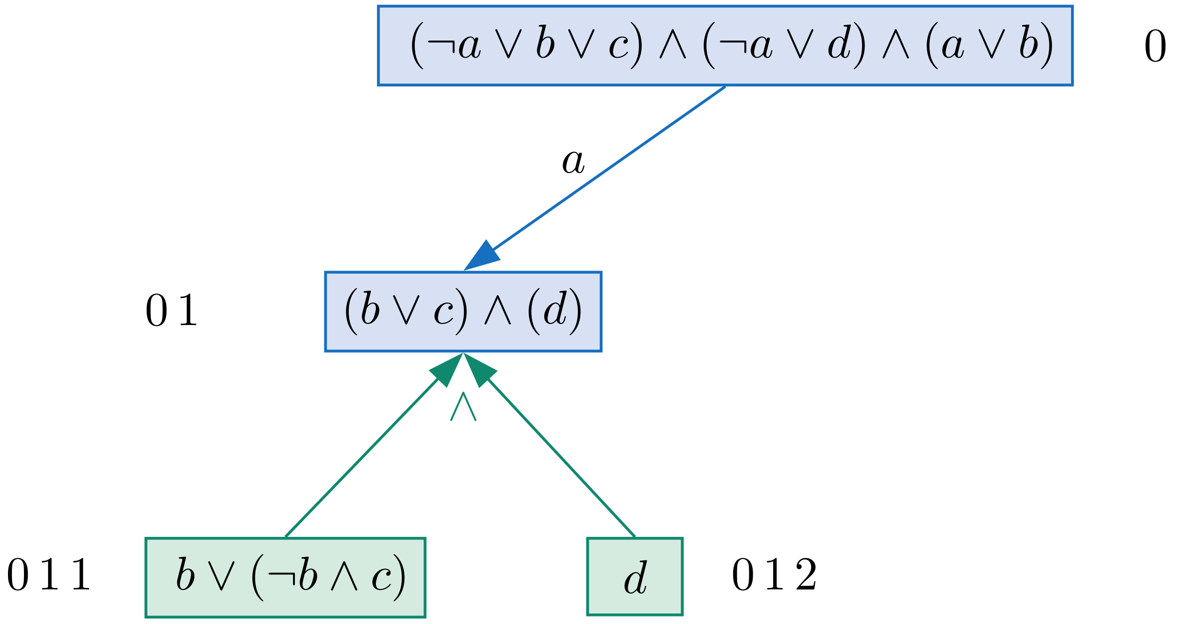
# Example

$$\begin{tikzcd}
 & & \boxed{(\neg a \vee b \vee c) \wedge (\neg a \vee d) \wedge (a \vee b)} \arrow[dl, "a"'] & 0 \\
0\,1 & \boxed{(b \vee c) \wedge (d)} & &
\end{tikzcd}$$

# Example


$(\neg a \lor b \lor c) \land (\neg a \lor d) \land (a \lor b)$
0
$a$
0 1
$(b \lor c) \land (d)$
$\land$
0 1 1
$(b \lor c)$
$(d)$
0 1 2

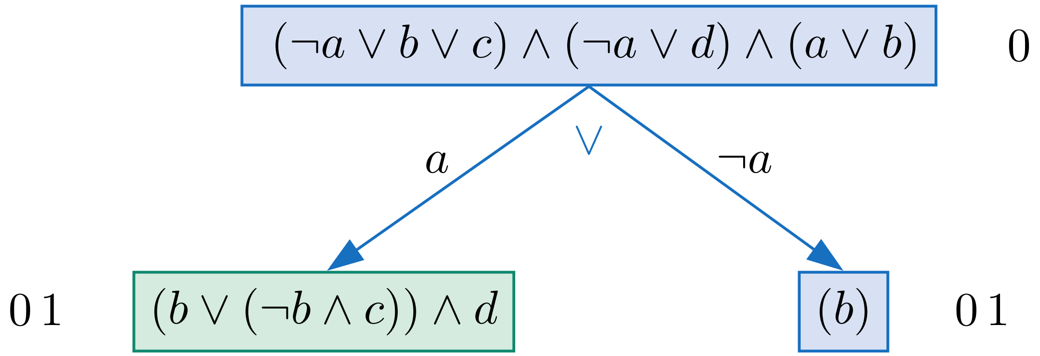
# Example


$(\neg a \lor b \lor c) \land (\neg a \lor d) \land (a \lor b)$
0
$a$
0 1
$(b \lor c) \land (d)$
$\land$
0 1 1
$b \lor (\neg b \land c)$
$d$
0 1 2

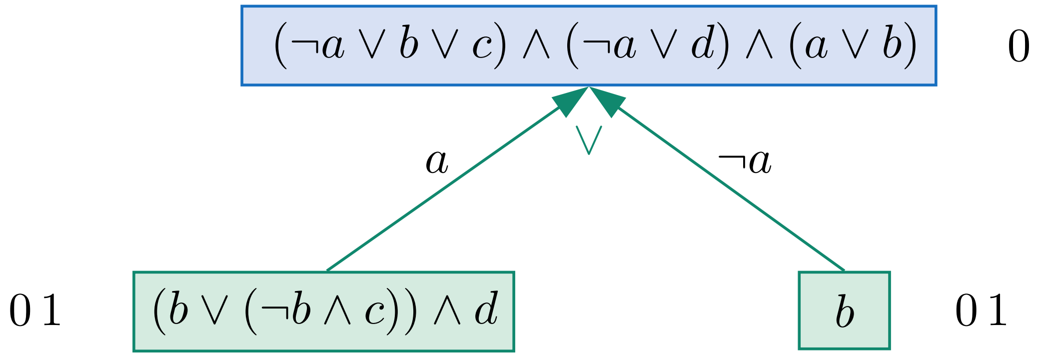
# Example

$$\begin{tikzcd}[column sep=small]
 & & \boxed{(\neg a \lor b \lor c) \land (\neg a \lor d) \land (a \lor b)} \arrow[dl, "a"'] & 0 \\
0\,1 & \boxed{(b \lor (\neg b \land c)) \land d} & &
\end{tikzcd}$$

# Example


$(\neg a \vee b \vee c) \wedge (\neg a \vee d) \wedge (a \vee b)$
0
$a$
$\vee$
$\neg a$
0 1
$(b \vee (\neg b \wedge c)) \wedge d$
$(b)$
0 1

# Example


$(\neg a \vee b \vee c) \wedge (\neg a \vee d) \wedge (a \vee b)$
0
$\vee$
$a$
$\neg a$
0 1
$(b \vee (\neg b \wedge c)) \wedge d$
$b$
0 1

# Example

$$\boxed{(a \wedge (b \vee (\neg b \wedge c)) \wedge d) \vee (\neg a \wedge b)} \quad 0$$

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

# Combining CDCL with Chronological Backtracking

| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $I$ | $\cdots$ | 4 | **5** | 30 | 47 | 15 | 18 | **6** | -7 | -8 | 45 | **9** | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

decision literal

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $I$ | $\cdots$ | 4 | **5** | **30** | **47** | **15** | **18** | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

block$(I, 4)$

# Combining CDCL with Chronological Backtracking

| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $I$ | $\cdots$ | 4 | **5** | **30** | **47** | **15** | **18** | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

slice$(I, 4)$

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $I$ | $\cdots$ | **4** | **5** | **30** | **47** | **15** | **18** | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

$$I_{\leqslant 4}$$

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 | ↯ |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

conflicting $\{$ -47, -17, -44 $\}$

conflict level 6

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

conflicting $\{$ -47, -17, -44 $\}$

learned $\{$ -30, -47, -18, 23 $\}$

jump level 4

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | 9 | 38 | -23 | 17 | 44 | -16 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |

conflicting { -47, -17, -44 }

learned { -30, -47, -18, 23 }

backtrack level 5

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 | **23** |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | **4** |

out of order

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $I$ | $\cdots$ | 4 | **5** | **30** | **47** | **15** | **18** | 6 | -7 | -8 | 45 | 23 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 4 |

$\mathsf{block}(I, 4)$

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $I$ | $\cdots$ | 4 | **5** | **30** | **47** | **15** | **18** | 6 | -7 | -8 | 45 | **23** |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 4 |

slice$(I, 4)$

# Combining CDCL with Chronological Backtracking

| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $I$ | $\cdots$ | 4 | 5 | 30 | 47 | 15 | 18 | 6 | -7 | -8 | 45 |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $I$ | $\cdots$ | **4** | **5** | **30** | **47** | **15** | **18** | 6 | -7 | -8 | 45 | **23** |
| $\delta$ | $\cdots$ | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 4 |

$I_{\leqslant 4}$

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $I$ | $\cdots$ | 18 | 6 | -7 | -8 | 45 | 23 | -38 | -9 | 10 | -11 | 13 | 16 | -17 | -25 | 42 | 12 | -41 |
| $\delta$ | $\cdots$ | 4 | 5 | 5 | 5 | 5 | 4 | 4 | 5 | 6 | 7 | 5 | 4 | 4 | 4 | 4 | 5 | 5 |

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $I$ | $\cdots$ | 18 | 6 | -7 | -8 | 45 | 23 | -38 | -9 | 10 | -11 | 13 | 16 | -17 | -25 | 42 | 12 | -41 |
| $\delta$ | $\cdots$ | 4 | 5 | 5 | 5 | 5 | 4 | 4 | 5 | 6 | 7 | 5 | 4 | 4 | 4 | 4 | 5 | 5 |

conflicting $\{$ 17, -42, -12 $\}$

# Combining CDCL with Chronological Backtracking

| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $I$ | $\cdots$ | 18 | 6 | -7 | -8 | 45 | 23 | -38 | -9 | 10 | -11 | 13 | 16 | -17 | -25 | 42 | 12 | -41 |
| $\delta$ | $\cdots$ | 4 | 5 | 5 | 5 | 5 | 4 | 4 | 5 | 6 | 7 | 5 | 4 | 4 | 4 | 4 | 5 | 5 |

conflicting { 17, -42, -12 }

backtrack level 4

# Combining CDCL with Chronological Backtracking

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\cdots$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| $I$ | $\cdots$ | 18 | 23 | -38 | 16 | -17 | -25 | 42 | -12 |
| $\delta$ | $\cdots$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |