Building small worlds in mathematical texts by formal and informal means

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1 Proof texts

Features and Theses

Features of the language of mathematics

- Proof texts are characterized by recursive nested structures (conditionals, case distrinctions, subproofs).
- Most proof texts make vast use of formal notation.

Theses regarding the language of mathematics

- Proof texts exhibit (nearly) all features of texts in other domains that make automatic interpretation hard, differing in degree.
  - The semi-formal language of mathematics has developped a lot of means to reduce ambiguity: reference by variables, bracketing, extraposition of quantifiers, explicit structuring of texts, ...
- Proof texts resemble narratives in small worlds.
  - They show a high degree of co-reference and of grouping referents into complexes.
  - The grouping is achieved by means of notation and by polysemy.

2 Small worlds

Reference Markers

- The number of objects mathematics talks about is big (transfinite) (models, discourse domains).
- “all natural numbers n” refers to a tranfinite number of objects.
- These objects are addressed by a single referential expression (referring to an indefinit/generic “object”).
- Inspired by DRT or the Naproche PRSs: Most (generalized) quantifiers (like “all natural numbers n”) introduce two *reference markers* into a semantic representation: the plural entity of *all* natural numbers and an indefinit number n.

Keeping worlds small: Complexes of reference markers

Complexes of reference markers

- In everyday domains reference markers are organized in complexes.
- a house: walls, roof, windows, doors, rooms etc.
- These complexes can be described as instances of *frames* (cf. e.g. Fisseni, Sarikaya, Schmitt, and Schröder (2019), Carl et al. (2021), and Fisseni, Sarikaya, and Schröder (2023))

Notational means

- AB, $\overline{AB}$, $\overrightarrow{AB}$, $|AB|$
- $f(x), f'(x), F(x)$
- $P = \{p_1, p_2, p_3, \ldots\}$

Givenness and activation of reference markers

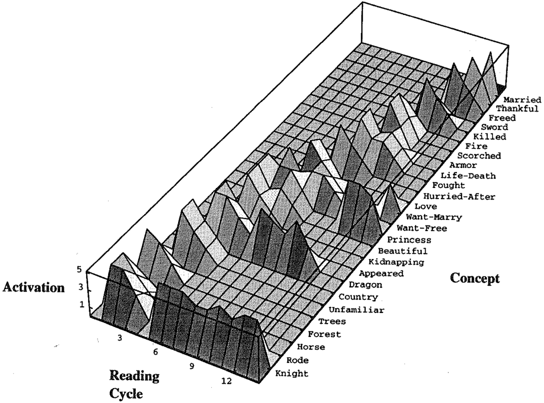
- Limited working memory, attention
- Referring expressions (indef. NP, def. NP, pronouns, zero reference)

Theoretical approaches

- Givenness (Prince (1981))
- Centering (Grosz, Joshi, and Weinstein (1995) , Walker, Joshi, and Prince (1998))
- Landscape model of reading (van den Broek, Risden, Fletcher, and Thurlow (1996), van den Broek, Young, Tzeng, and Linderholm (1999)), „activation“

Landscape model of reading

- Mentions of, references to concepts activate these concepts
- Activation fades out after mention.
- Strong correlation between memorizing concepts and overall activation in a text.
- Strong correlation between co-memorizing concepts and similarity of activation patterns in a text



(from: Van den Broek, Risden, Fletcher, and Thurlow 1996, p. 173)

Activation of concepts, extension for mathematics

- 5: Explicit mention; *objects referenced by compound nouns and complex math. notation*
- 4: pronominal anaphor, needed for coherence
- 3: *objects referenced by constituents of compound nouns and complex math. notation*
- 2: inferred from the context
- Activation halves in subsequent sentences without a renewal of the concept.

The knight story

A young knight rode through the forest.

The knight was unfamiliar with the country. Suddenly, a dragon appeared.

The dragon was kidnapping a beautiful princess. The knight wanted to free her.

The knight wanted to marry her.

The knight hurried after the dragon.

They fought for life and death.

Soon, the knight's armor was completely scorched. At last, the knight killed the dragon.

He freed the princess.

The princess was very thankful to the knight.

She married the knight.

(Van den Broek, Risden, Fletcher, and Thurlow 1996, p. 170)

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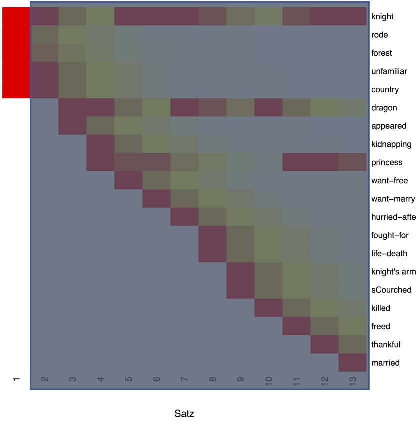

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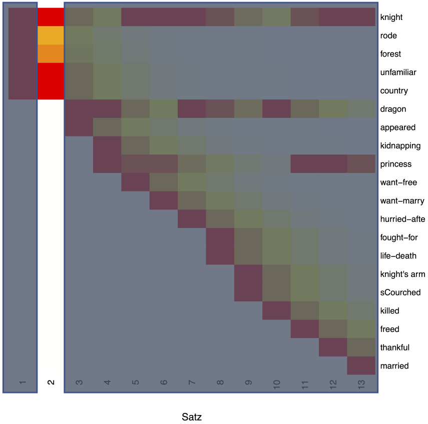

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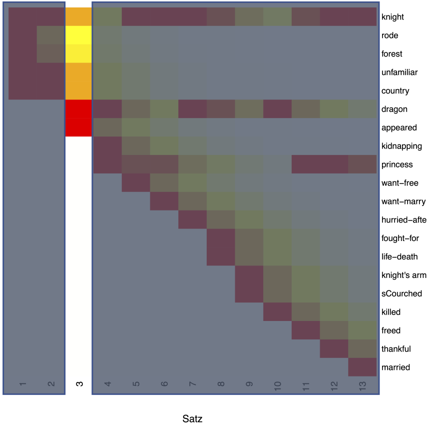

The knight story (4)

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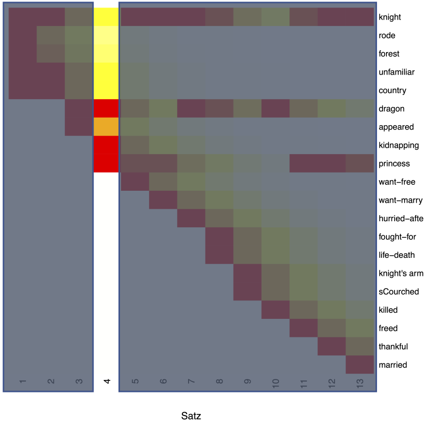
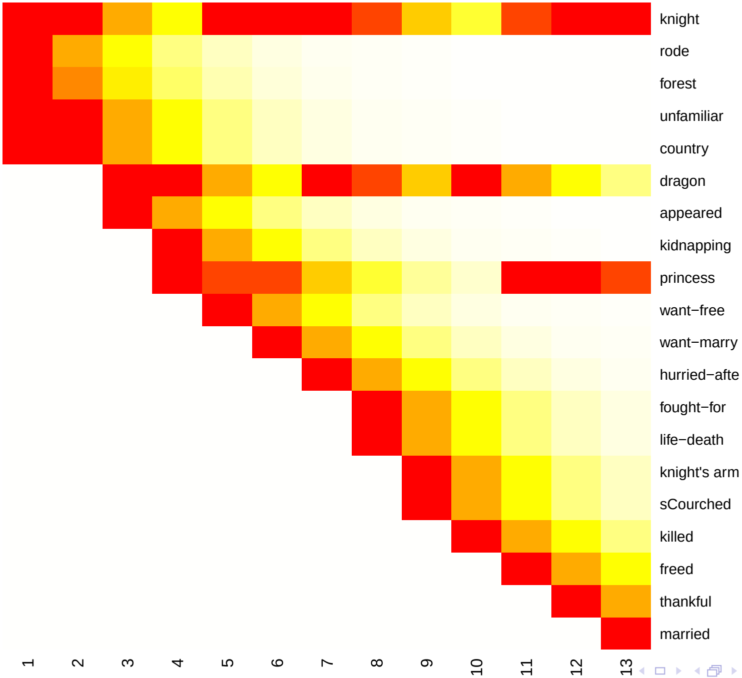
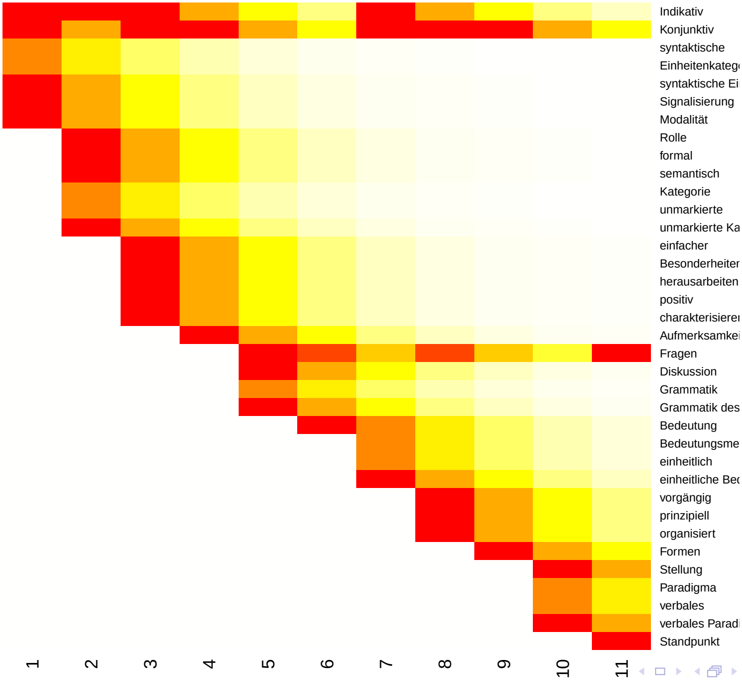


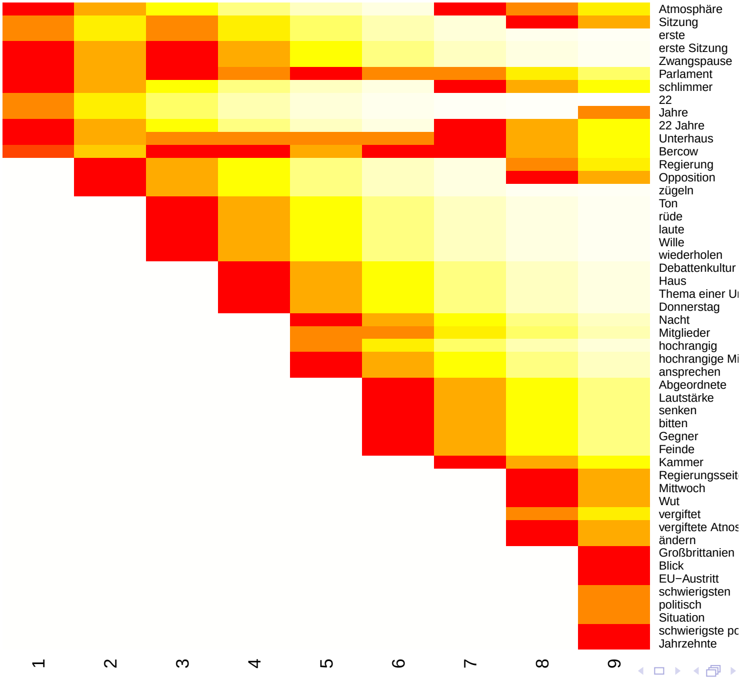
Activation landscape of the knight story



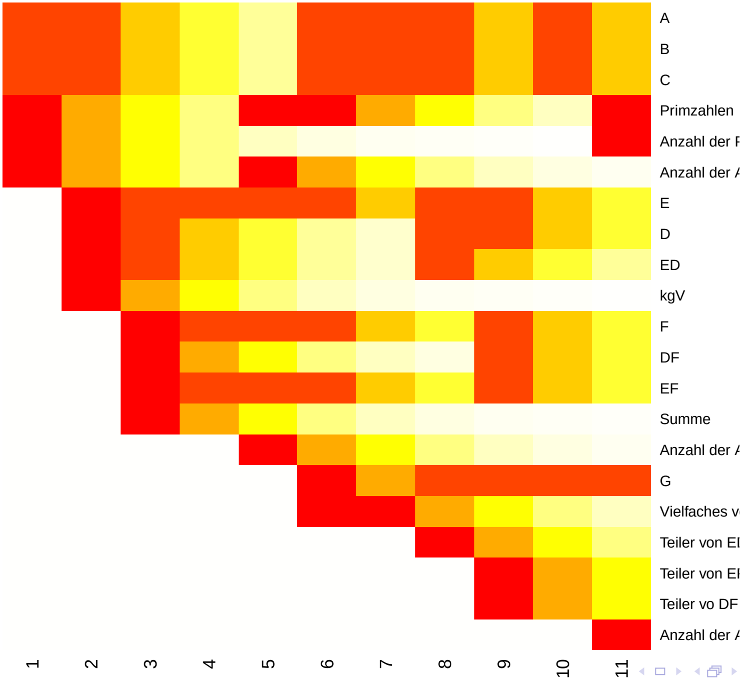
Activation landscape of a non-fiction text (linguistics)



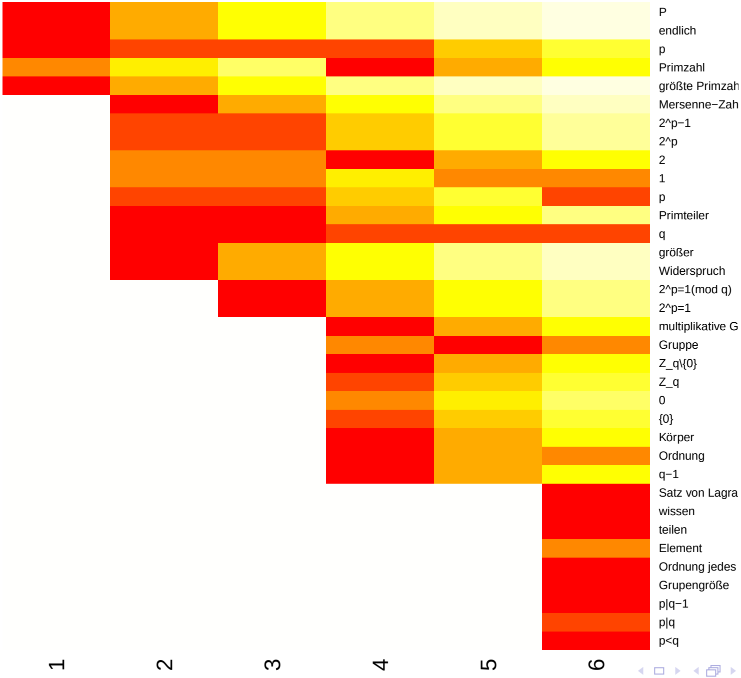
Activation landscape of a newspaper text



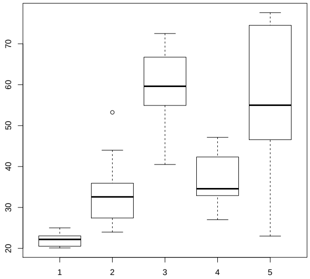
Euclid’s proof of the inifinity of prime number



Aigner/Ziegler 2010, 4th proof of the infinity of prime numbers

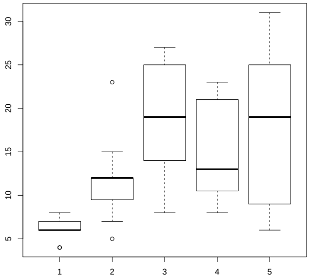


Activation sums per sentence



1 Knight story
2 Linguistics
3 Newspaper
4 Euclid
5 Aigner/Ziegler

Activation sums per sentence / sentence lengths



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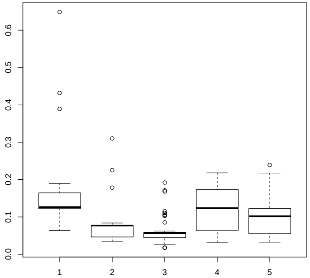
Activation sums of concepts (relative to text lengths)

distribution of the line sums of the heatmaps / text length



1 Knight story

2 Linguistics

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4 Euclid

5 Aigner/Ziegler

Reference structure of nathematical text

Mathematical texts show a high level of co-reference (reusing of discourse markers and their complexes)

3 Ambiguous reference to complexes

Semantic underdetermination

Semantic underdetermination: any feature of NL that prevents a full truth-conditional interpretation of a NL expression without regarding its context

NL sources of semantic underdetermination

- Vagueness
- Argumentative gaps
- Metaphor of time
- Ambiguity
- Explicature, implicature
- Presuppositions

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NL sources of semantic underdetermination

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- Ambiguity
  - Homonymy
  - Polysemy
  - Syntactic ambiguity
  - Scope ambiguity
- Explicature, implicature
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Scope ambiguities

Scope ambiguities

- Scope of assuptions
- Scope of connectives

(1) A and B or C

(1a) (A and B) or C

(1b) A and (B or C)

- Quantifier scope

Quantifier scope ambiguities

Scope ambiguities

(2) All students of our university should read a book.

$\forall\exists$, $\exists\forall$?

Quantifier scope ambiguities

Scope ambiguities

(2) All students of our university should read a book.

$\forall\exists$, $\exists\forall$?

(2') At the UDE Olivias Garten by Alina Bremer was chosen in the program of the Stifterverband "Eine Uni ein Buch".

-> (2): $\exists\forall$!

Quantifier scope ambiguities in mathematics

Scope ambiguities

(3) Some element of any nonempty set S is not a subset of S.
- ∀ non-empty set ∃ element

(4) Any points belong to some line.

(examples by Andrei Paskevitch)

Polysemy

Polysemy: signs (words, phrases, sentences) having multiple *related* (vague!) readings.

Types of polysemy

- non-linear
  - autohyponymy, where the basic sense leads to a specialised sense (from “drinking (anything)” to “drinking (alcohol)”)
  - automeronymy, where the basic sense leads to a subpart sense (from “door (whole structure)” to “door (panel)”)
  - autohyperonymy or autosuperordination, where the basic sense leads to a wider sense (from “(female) cow” to “cow (of either sex)”)
  - autoholonymy, where the basic sense leads to a larger sense (from “leg (thigh and calf)” to “leg (thigh, calf, knee and foot)”)
- linear
  - metonymy, where one sense “stands for” another (from “hands (body part)” to “hands (manual labourers)”)
  - metaphor, where there is a resemblance between senses (from “swallowing (a pill)” to “swallowing (an argument)”)
  - other construals (for example, from “month (of the year)” to “month (30 days)”)

(Cruse, D Alan (2000). “Contextual variability”. Meaning in Language. Oxford University Press)

- plural readings

Metonymy

Features of metonymy

- Examples
  - “height”: dimension of measure, geographical place
  - “school”: institution, building
  - “London”: town, British government
- readings often ontologically/categorically different
- connected by a “contiguous” relation
- a small set of contiguous relations

Metonymy: contiguous relations

Contiguous relations

- Institution -> place: “university”
- Place -> institution (toponymy): “London”, “the Wall Street”
- Containment: “drink a glass”
- Physical item, place, or body part: “many hands make light work”
- Tools, instruments: “press”
- Product for process: “the book was nightmare”
- Synecdoche: “meals on wheels”

Metonymy in mathematics

Metonymy in mathematics

- Result–structure relation: "sum", "product"

(5) The sum of x and y is not divible by 3.

(6) r is the square-free part of the product of the numbers in M.

- "the product ..." does not necessarily refer to a concrete syntactic structure of the product
  - $2 \cdot 4 \cdot 5 \cdot 9$ -> square-free part: $2 \cdot 5$
  - refers to an eqivalence class of syntactic structures allowing associative and commutative transformations
- reference to the concrete syntax is also possible: "the sum $a + b + c + d$ ...", "take the first summand"

Plurals

Plurals

- implicit introduction of plural entities, i.e. entities like sets, sequences etc. comprising more than one member (see Nouwen 2015, Cramer/Schröder 2012 and Schröder in print)

(3) a, b and c are P.

- Plural entity {a,b,c} can be referenced anaphorically: “they”

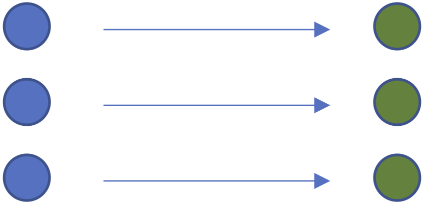
Common plural readings

Plural readings

- distributive
- collective
- cumulative

Distributive plurals

(7) Three musicians of the chamber orchestra played a string instrument.



Distributive plurals

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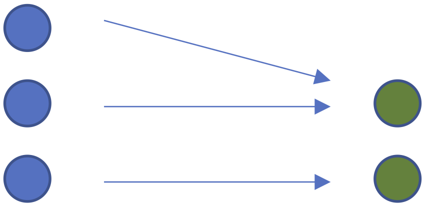
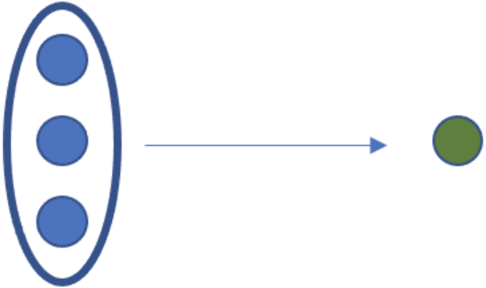


(reference to sorts of string instruments)

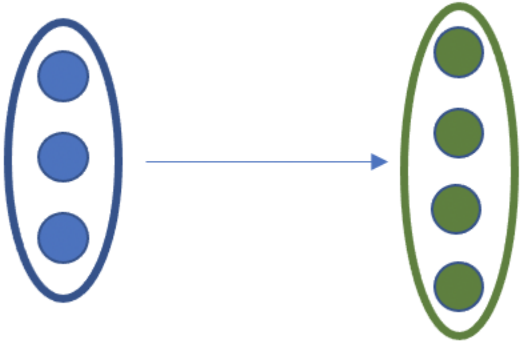
Collective plurals

(8) Three men carried a piano.



Cumulative plurals

(9) Three of the guests drank four bottles of wine.



Statistics:

(10) 10% of the German population hold 63% of the national wealth.

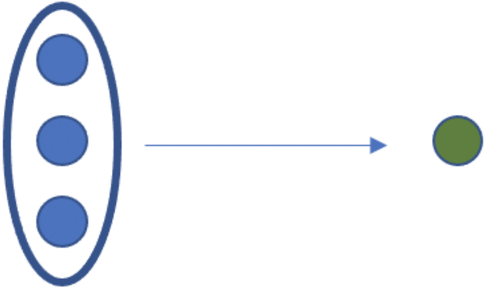
Distributive plurals

(11) $n_1, n_2, n_3, \ldots$ are divisible by 3.

Each of them!

Collective plurals

(12) Let $p_1, p_2, p_3, \ldots$ be a sequence of primes in increasing order ...



Some predicates trigger collective readings (here: “sequence”).

Symmetric predicates and collective readings

Symmetric predicates

$aSb \rightarrow bSa$ - “equals”, “equaivalent”, “congruent”, ... - “different”, “unequal”, “incongruent”, ...

Symmetric predicates

(13) a is different from b.

(14) a and b are different.

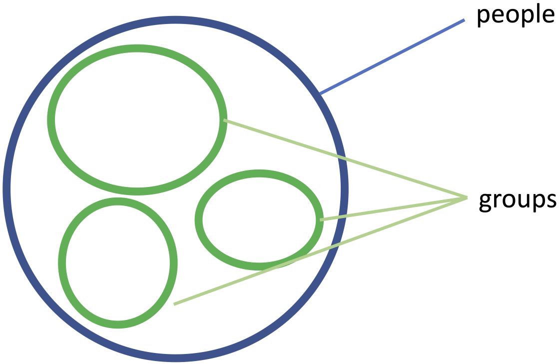
From symmetric two-place relations one-place predicates applicable to plural entities can be derived.

$aSb \Rightarrow S'(\{a,b\})$

Complexities in plural ambiguities

(15) In each group there were different people.

(16) Each group consisted of different people.

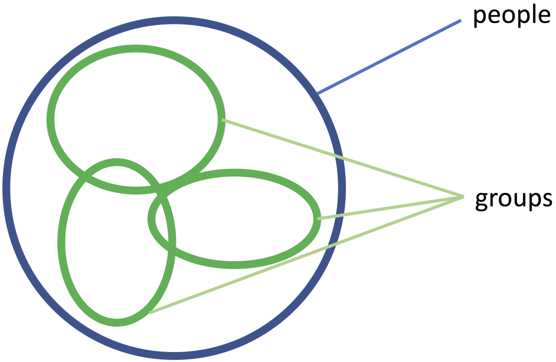


Complexities in plural ambiguities

(17) In each group there were different people.

- Negation of: In each group were the same people.

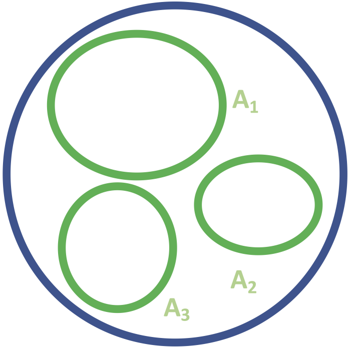
(18) Each group consisted of different people.

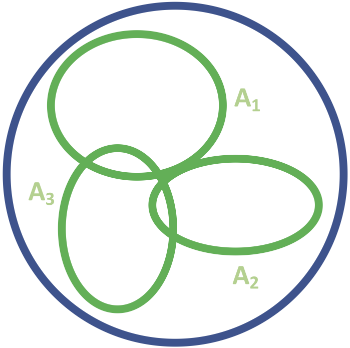


Complexities in plural ambiguities

(19) The sets A_1, A_2, A_3 consist of different members.

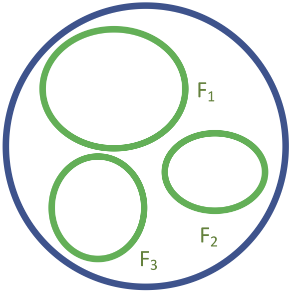
(20) A_1, A_2, A_3 are sets with different members.



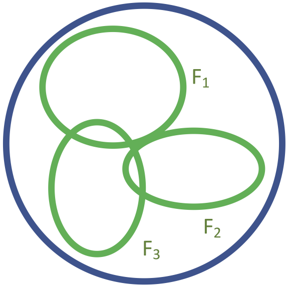


Plural ambiguities in real life

(23) Every a_n is thus a product of different small primes [...]



The plural entities of the factors of a_n are pairwise disjunct.



The plural entities (aos) of the factors of a_n are pairwise distinct.

What are the plural entities in this case formally? - not sets, we need multiple occurrences of primes factors - ordering is irrelevant

–> alphabetically ordered sequence (aos)?

F_n: (plural entity (aos) of) factors of the product a_n

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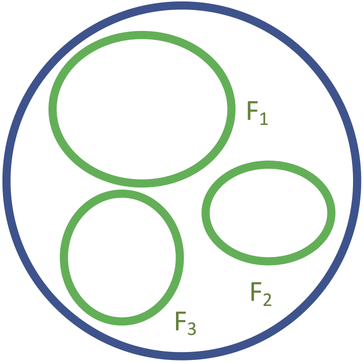

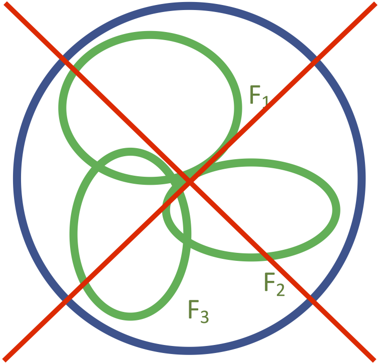

The plural entities (aos) of the factors of a_n are pairwise disjunct.

~~The plural entities (aos) of the factors of a_n are pairwise distinct.~~
Trivial if the a_ns are pairwise unequal.

F_n: (plural entity (aos) of) factors of the product a_n

Plural ambiguities in real contexts

Let us now look at N_s. We write every $n \leq N$ which has only small prime divisors in the form $n = a_n b_n^2$, where a_n is the square-free part. Every a_n is thus a product of *different* small primes, and we conclude that there are precisely 2^k different square-free parts. Furthermore, as $b_n \leq \sqrt{n} \leq \sqrt{N}$, we find that there are at most $\sqrt{N}$ different square parts, and so

$$N_s \leq 2^k\sqrt{N}.$$

- ~~The plural entities (aos) of the factors of a_n are pairwise distinct.~~
- ~~The plural entities (aos) of the factors of a_n are pairwise disjunct.~~
- *The members of the aos of factors of a_n are pairwise different for each n.*
- Obviously, for each n the sequence of factors of a_n as well as the set of all such sequences are semantically available for “different”.

Groups of people and products

How can the different interpretations for groups and products be explained?

(15) In each group there were different people.

(16) Each group consisted of different people.

(17) Every a_n is thus a product of different small primes [...]

- The plural expression "people"/"small primes" introduces a plural entity P.
- For each group g_n / each a_n a plural entity P_n is implicitly introduced.
- Therefore the plural entity $\mathscr{P}$ of all P_ns becomes semantically available.
- "different′" can be applied to every P_n or to $\mathscr{P}$.
- Application to P_n is not informative for groups (= sets) of people, but for products (treated as sequences) of factors. Therefore, the application of "different′" to the P_ns is ruled out for groups of people, but not for products.
- The squarefreeness implies uniqueness of the factors. Therefore, pairwise unequality of the factors (as members of P_n) is under discussion.
- This raises the expectation that "different′" is to be applied to the P_ns.

Polysemy in mathematics

- It is quite implausible that polysemous readings are systematically evaluated when reading a proof.
- Readers usually are guided by the context to the intended interpretation.
- goal-/expectation-driven -> frames

Conclusions

Small worlds

- Worlds are kept small in proof texts.
    - High level of co-reference.
    - Grouping of reference markers in complexes.
        - by notational means
        - by polysemy

Polysemy

- In case of polysemy: Mathematical texts make related interpretations available.
- Scaffold for a goal-/expectation-driven construction of ontologically and quantficationally adequate readings from contextual information.

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