GeoCoq: a library for foundations of geometry

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Outline

Overview

Axiom systems

Tarski, Hilbert, Euclid or analytic approach.
Multiple models/counter-models.
Many versions of the parallel postulate.

Some numbers

- 150 kloc
- 4700 lemmas
- 900 definitions

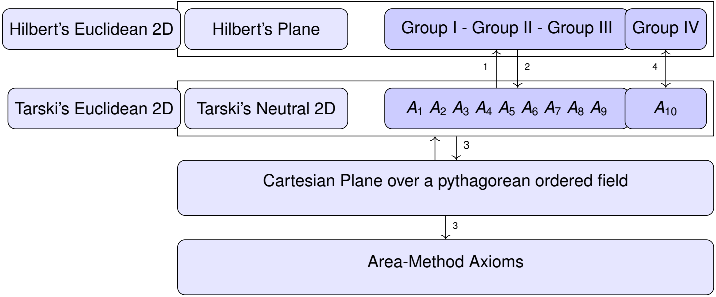
Automation

- Area method
- Algebraic methods

Some well-known theorems

- Pappus, Desargues, Pythagoras, Thales
- Midpoint theorem
- SAS, SSS, ...
- Quadrilaterals
- Triangle centers
- Euler line
- ...

Overview of the axiom systems



[1] Gabriel Braun and Julien Narboux (Sept. 2012). "From Tarski to Hilbert". English. In: Post-proceedings of Automated Deduction in Geometry 2012. Vol. 7993. LNCS

[2] Gabriel Braun, Pierre Boutry, and Julien Narboux (June 2016). "From Hilbert to Tarski". In: Eleventh International Workshop on Automated Deduction in Geometry. Proceedings of ADG 2016

[3] Pierre Boutry, Gabriel Braun, and Julien Narboux (2019). "Formalization of the Arithmetization of Euclidean Plane Geometry and Applications". In: Journal of Symbolic Computation 98

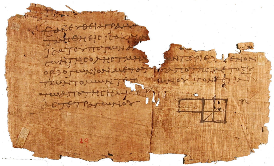
[4] Pierre Boutry et al. (2017). "Parallel postulates and continuity axioms: a mechanized study in intuitionistic logic using Coq". In: Journal of Automated Reasoning

Outline

1. Overview of the GeoCoq Library
   - Two formalizations of the Elements
   - Arithmetization of Geometry
   - Continuity axioms
   - 34 parallel postulates
2. What features GeoCoq uses?
3. GeoCoq ported to other proof assistants

The Elements

- A very influential mathematical book (more than 1000 editions).
- First known example of an axiomatic approach.



Book 2, Prop V, Oxyrhynchus Papyri (year 100)

Euclid

First project

- Joint work with Charly Gries and Gabriel Braun.
- Mechanizing proofs of Euclid's **statements**.
- Not Euclid's proofs!
- Trying to minimize the assumptions:
  - Parallel postulate,
  - Elementary continuity,
  - Archimedes' axiom.

Second project

- Joint work with Michael Beeson and Freek Wiedijk[5].
- Formalizing Euclid's **proofs**.
- A not minimal axiom system (current effort, together with Prunelle Colin, to remove "equal figures" axioms[6]).
- Filling the gaps in Euclid.

[5] Michael Beeson, Julien Narboux, and Freek Wiedijk (2019). "Proof-checking Euclid". In: Annals of Mathematics and Artificial Intelligence 85.2-4

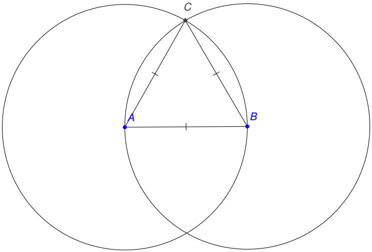
[6] Michael Beeson (July 2022). "On the notion of equal figures in Euclid". In: Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry 64.3

Example

Proposition (Book I, Prop 1)

Let A and B be two points, build an equilateral triangle on the base AB.

Proof: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ the circles of center A and B and radius AB. Take C at the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$. The distance AB is congruent to AC, and AB is congruent to BC. Hence, ABC is an equilateral triangle.

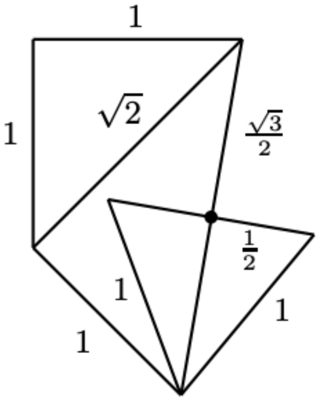


Book I, Prop 1

In the spirit of *reverse mathematics*, we proved two statements:

1. Assuming no continuity, but the parallel postulate (solving a challenge proposed by Michael Beeson[7]).
2. Assuming circle/circle continuity, but not the parallel postulate (trivial).

Victor Pambuccian has shown that these assumptions are minimal[8].



[7] Michael Beeson (2013). "Proof and Computation in Geometry". In: Automated Deduction in Geometry (ADG 2012). Vol. 7993. Springer Lecture Notes in Artificial Intelligence

[8] Victor Pambuccian (1998). "Zur Existenz gleichseitiger Dreiecke in H-Ebenen". In: Journal of Geometry 63.1

A proof assuming no continuity

```
Section Book_1_prop_1_euclidean.
Context `{TE:Tarski_2D_euclidean}.

Lemma prop_1_euclidean :
  forall A B,
  exists C, Cong A B A C /\ Cong A B B C.
Proof. ... Qed.

End Book_1_prop_1_euclidean.
```

A proof assuming circle/circle continuity

```
Section Book_1_prop_1_circle_circle.
Context `{TE:Tarski_2D}.

Lemma prop_1_circle_circle :
  circle_circle_bis ->
  forall A B,
  exists C, Cong A B A C /\ Cong A B B C.
Proof.
intros H A B.
destruct (H A B B A A B) as [C [HC1 HC2]]; Circle.
exists C.
unfold OnCircle in *.
split; Cong.
Qed.

End Book_1_prop_1_circle_circle.
```

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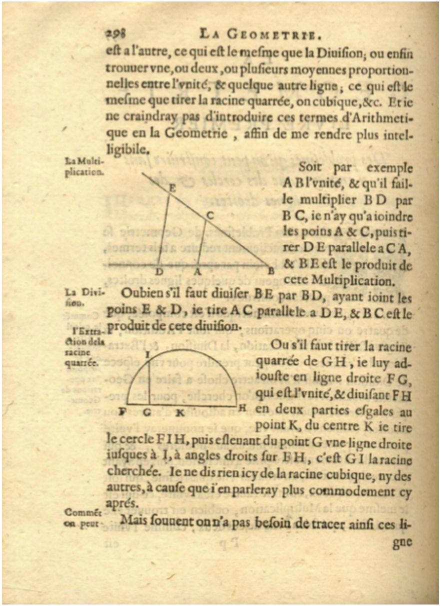
Arithmetization of Geometry

René Descartes (1925).
La géométrie.



eſt a l'autre, ce qui eſt le meſme que la Diuiſion; ou enfin trouuer vne, ou deux, ou pluſieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui eſt le meſme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligible.

La Multiplication.

Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.

La Diuiſion.

Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

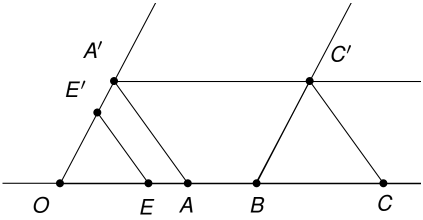
l'Extraction de la racine quarrée.

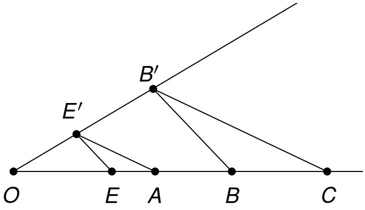
Ou s'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au point K, du centre K ie tire le cercle F I H, puis eſleuant du point G vne ligne droite iuſques à I, à angles droits ſur F H, c'eſt G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cauſe que i'en parleray plus commodement cy aprés.

Commét on peut

Mais ſouuent on n'a pas beſoin de tracer ainſi ces ligne

Addition and multiplication





Automation

This is not a theorem about polynomials[9]:

```
Section T18.
Context `{T2D:Tarski_2D}.
Context `{TE:@Tarski_euclidean Tn TnEQD}.

Lemma centroid_theorem : forall A B C A1 B1 C1 G,
  Midpoint A1 B C ->
  Midpoint B1 A C ->
  Midpoint C1 A B ->
  Col A A1 G ->
  Col B B1 G ->
  Col C C1 G \/ Col A B C.
Proof.
intros A B C A1 B1 C1 G; convert_to_algebra; decompose_coordinates.
intros; spliter. express_disj_as_a_single_poly; nsatz.
Qed.

End T18.
```

[9] Benjamin Grégoire, Loïc Pottier, and Laurent Théry (2011). "Proof Certificates for Algebra and Their Application to Automatic Geometry Theorem Proving". In: Post-proceedings of Automated Deduction in Geometry (ADG 2008). Vol. 6301. Lecture Notes in Artificial Intelligence

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Hilbert's line completeness

Axiom V.2: "An extension (An extended line from a line that already exists, usually used in geometry) of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-III and from V-1 is impossible."

> *"Hilbert's own completeness axiom, added in other editions as V-2, takes the somewhat awkward form of requiring that it be impossible to properly extend the sets and relations satisfying the other axioms so that all the other axioms still hold."*

– Martin 1998, p. 175

Formalization in Rocq

We need to quantify over models of other axioms[10]:

```
Section Completeness.
Context `{Tn:Tarski_neutral_dimensionless}.

Definition completeness_for_planes := forall
  (Tm: Tarski_neutral_dimensionless)
  (Tm2 : Tarski_neutral_dimensionless_with_decidable_point_equality Tm)
  (M : Tarski_2D Tm2)
  (f : @Tpoint Tn -> @Tpoint Tm),
  @archimedes_axiom Tm ->
  extension f ->
  forall A, exists B, f B = A.

End Completeness.
```

[10] Charly Gries, Julien Narboux, and Pierre Boutry (Jan. 2019). “Axiomes de continuité en géométrie neutre : une étude mécanisée en Coq”. In: Journées Francophones des Langages Applicatifs 2019. Acte des Journées Francophones des Langages Applicatifs (JFLA 2019)

Algebra/Geometry

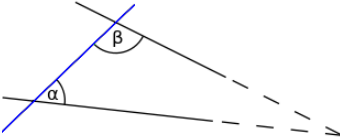
Geometry	Algebra
	Pythagorean ordered field[11]
circle/line continuity	Euclidean ordered field[12]
FO Dedekind cuts	real closed field[13]
Dedekind cuts	reals

[11] The sum of squares is a square.
[12] Every non-negative is a square.
[13] Euclidean ordered and every polynomial of odd degree has at least one root in the field.

Outline

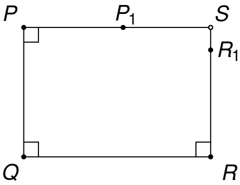
Euclid 5th postulate

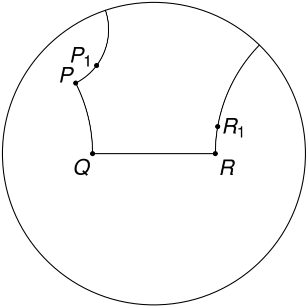
"If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough."



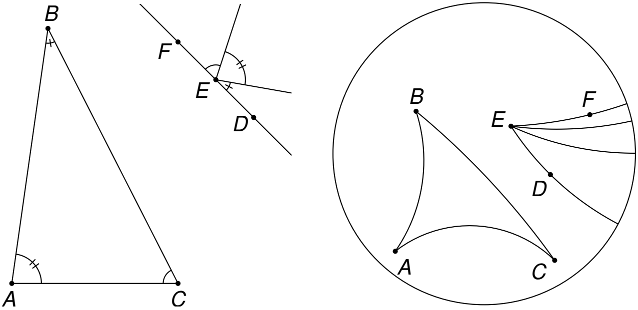
Bachmann's Lotschnittaxiom

If $PP_1 \perp PQ$, $PQ \perp QR$ and $QR \perp RR_1$ then PP_1 and RR_1 meet.

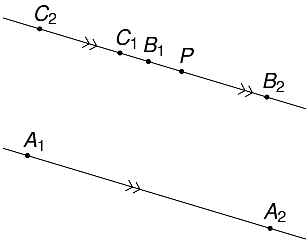


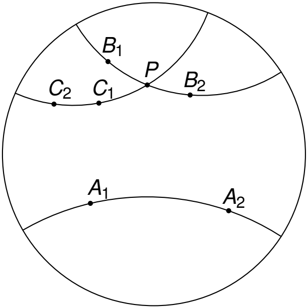


Triangle postulate

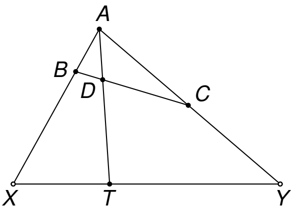


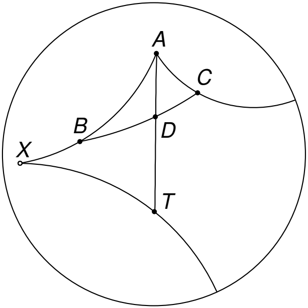
Playfair's postulate



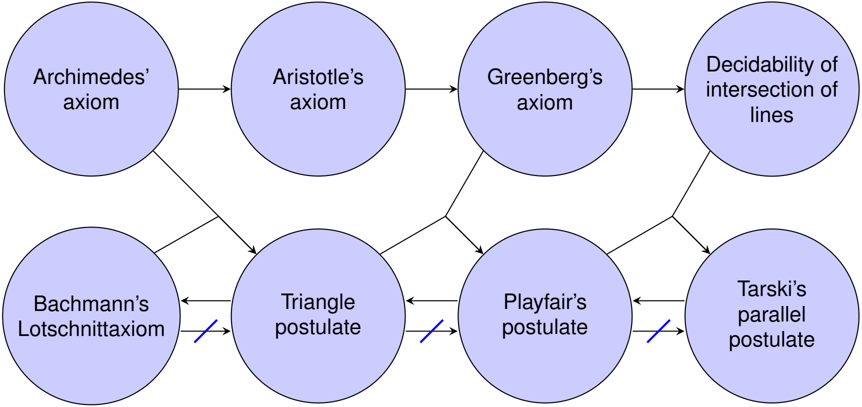


Tarski's postulate



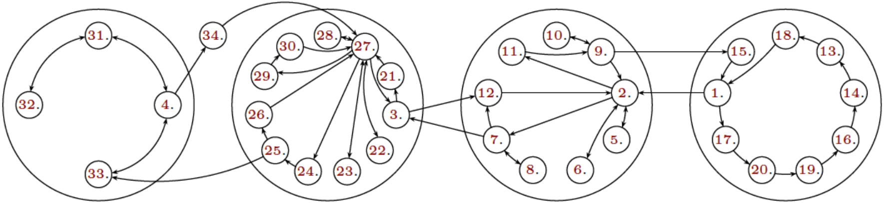


Four groups



Archimedes' axiom
Aristotle's axiom
Greenberg's axiom
Decidability of intersection of lines
Bachmann's Lotschnittaxiom
Triangle postulate
Playfair's postulate
Tarski's parallel postulate

Sorting 34 postulates



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[14] Pierre Boutry et al. (2017). "Parallel postulates and continuity axioms: a mechanized study in intuitionistic logic using Coq". In: Journal of Automated Reasoning

Outline

First-order vs Higher-order logic in GeoCoq

- Formally all proofs are higher-order: forall predicates “cong” and “bet” verifying the axioms, if ... then ...
- But many proofs a locally first-order (if we assume the axioms to be in the context).
- Tarski’s axiom system is meant to be expressed in FOL.

Use of higher-order logic

- Meta-theoretical results.
- In the proof of Pappus' theorem[15], the following concept is used: equivalence classes of congruent segments.
- Continuity axioms.

[15] Gabriel Braun and Julien Narboux (Feb. 2017). "A synthetic proof of Pappus' theorem in Tarski's geometry". In: Journal of Automated Reasoning 58.2

Outline

Constructive or classical logic?

Intuitionistic logic:

- Assuming: $\forall A, B$: Points, $A = B \vee A \neq B$,
- We proved: excluded middle for all other predicates[16].

[16] Pierre Boutry et al. (July 2014). "A short note about case distinctions in Tarski's geometry". In: Proceedings of the 10th Int. Workshop on Automated Deduction in Geometry. Vol. TR 2014/01. Proceedings of ADG 2014

Constructive or classical logic?

Intuitionistic logic:

- Assuming: $\forall A, B$: Points, $A = B \vee A \neq B$,
- We proved: excluded middle for all other predicates[16], except line intersection!

[16] Pierre Boutry et al. (July 2014). "A short note about case distinctions in Tarski's geometry". In: Proceedings of the 10th Int. Workshop on Automated Deduction in Geometry. Vol. TR 2014/01. Proceedings of ADG 2014

Outline

Towards Automating the Constructivization of GeoCoq

- We wish to replace the $\forall A, B$: Points, $A = B \lor A \neq B$ assumption with $\forall A, B$: Points, $\neg A \neq B \Rightarrow A = B$[17] following Beeson[18].
- We are currently focusing on the "wholesale importation of proofs of negative theorems[19] from classical to constructive geometry".
- We can mimic classical reasoning as long as we are proving a *stable* formula: for example, we can reason based on a modified *modus ponens*

$$\textit{stable } B \Rightarrow \neg\neg A \Rightarrow (A \Rightarrow B) \Rightarrow B.$$

- To automate the modification we, together with Alexandre Jean and Nicolas Magaud, rely on the "Rocq-ditto" framework (see "Optimization of Rocq proof scripts" from Nicolas Magaud).

[17] We say that the equality of points is *stable*.

[18] Michael Beeson (2015). "A constructive version of Tarski's geometry". In: Annals of Pure and Applied Logic 166.11

[19] Formulas not containing ∃ or ∨.

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   - Automatically: using Dedukti
   - Manually: to Isabelle and Lean
   - The ADG-Lib initiative

GeoCoq/Euclid in Dedukti

Formalization of Book 1 of Euclid's Elements: 238 lemmas, 20 kloc (15% of GeoCoq).

Features: no inductive, no fixpoint, no reflexivity, first-order proofs, simple tactics.

Yoan Géran has exported our formalization of Euclid/Book 1 to: Rocq, HOL-Light, Lean, Matita, PVS and Open Theory:

The (compressed) size of the translated proofs are multiplied by 10 (Lean, Matita, Rocq), 25 (Hol-Light) and 50 (PVS).

GeoCoq/Tarski in Dedukti

Together with Yoan Géran: formalization the first twelve chapters of SST[20]: 1398 lemmas, 50 kloc (33% of GeoCoq).

This translation to an encoding of First-Order Logic required:

- modifications of GeoCoq to replace the use of reflexives tactics with proofs obtained by the Larus prover[21],
- some reverse mathematics in Dedukti[22].

The (compressed) size of the translated proofs are multiplied by less than 2 (Lean, Rocq).

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[20] Wolfram Schwabhäuser, Wanda Szmielew, and Alfred Tarski (1983). Metamathematische Methoden in der Geometrie.

[21] Predrag Janicic and Julien Narboux (2022). "Theorem Proving as Constraint Solving with Coherent Logic". In: J. Autom. Reason. 66.4

[22] `https://gitlab.crans.org/geran/dkpltact`

Outline

GeoCoq ported to Isabelle: IsaGeoCoq

Roland Coghetto started to port GeoCoq to Isabelle.

First version is available in the AFP since 2021 (22 kloc):

The second version is in preparation:

It contains 2850 lemmas, 18 locales and 92 kloc making it one of the largest Isabelle contributions (roughly 75% of GeoCoq).

GeoCoq/Tarski ported to Lean

Bhavik Mehta ported 448 lemmas of GeoCoq/Tarski to Lean (roughly 30% of GeoCoq/Tarski, 10% of GeoCoq):

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The ADG-Lib initiative

Initiated by Pedro Quaresma, Predrag Janičić, Julien Narboux, Zoltán Kovács, Anna Petiurenko, Nuno Baeta.

Encouraged by the success of the SMT-Lib initiative, which significantly advanced the field of SMT.

We set similar goals for the ADG-Lib initiative to foster advances in the field of Automated Deduction in Geometry.

- Provide a rigorous description of the axiom systems used by the different geometry, automated and interactive theorem provers.
- Propose a common input and output language for geometry theorem provers.
- Establish and make available to the research community a large library of geometric problems.

The ADG-Lib initiative

At the WG2-GEO25 meeting, we discussed the preliminary ADG-Lib signature.

The plan is to port the GeoCoq library of formal geometry proofs to ADG-Lib, when done, this will be version 1.0 of ADG-Lib.

Conclusions

- GeoCoq is the only formalization of geometry that goes up to the arithmetization and connects to automation using algebraic methods.
- While formalizing old results about geometry with a new tool (the proof assistant), new discoveries were made:
  - Using constructive logic, we get a finer classification of parallel postulates.
  - This inspired a new syntactic proof of the independence of the parallel postulate.
- The port to other proof assistants (Isabelle is our next target) opens the perspective of aligning axiom systems for hyperbolic geometry[23] with other AFP entries:
  - Poincaré Disk model by Danijela Simić, Filip Marić and Pierre Boutry,
  - The Independence of Tarski's Euclidean Axiom by Timothy James McKenzie Makarios.

[23] Pierre Boutry, Stéphane Kastenbaum, and Clément Saintier (Jan. 2024). "Towards an Independent Version of Tarski's System of Geometry". In: Electronic Proceedings in Theoretical Computer Science 398

In memory of Gilles Dowek.

Thank you for your attention.