Christian Merten

Utrecht University

Sep 15, 2025

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- General construction: Spec associates an affine scheme to any commutative ring.
- Affine schemes can be completely understood via the study of commutative rings.
- A scheme is a geometric object, that locally looks like an affine scheme.



#### Outline

A bit of history

Library overview

Definition of schemes

Reduction to affine schemes

Future work

2003 Formalization of affine schemes in Rocq (formerly Coq) by Chicli.

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#### A word on Lean and mathlib

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- Lean is a dependently typed interactive theorem prover, initially developed by Leonardo de Moura at Microsoft Research and since 2023 mainly developed by the Lean FRO.
- Mathlib is a user-maintained mathematical library for Lean, covering a broad range of mathematics.

#### Attributions

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- Angdinata and Xu are the driving forces behind elliptic curves.
- The schemes library has recently been mainly developed by Andrew Yang and C.M.

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- Projective space

#### Definition of Schemes

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- To introduce a scheme, we do: variable (X : Scheme)
- Main reason: many arguments in algebraic geometry use category theoretical tools.

## (Un)bundled geometric objects

• The differential geometry library follows the unbundled design:

```
variable {k : Type} [NontriviallyNormedField k]
  {E : Type} [NormedAddCommGroup E] [NormedSpace k E]
  {H : Type} [TopologicalSpace H] {I : ModelWithCorners k E H}
  {M : Type} [TopologicalSpace M] [ChartedSpace H M]
  {n : WithTop N∞} [IsManifold I n M]
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• Main advantage: A manifold is automatically also a topological space and every result immediately applies.

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- To reason about schemes, one reduces the problem to affine schemes and then solves the resulting commutative algebra problem.

Pour démontrer (i), notons que la question est locale sur X; on peut donc se restreindre au cas où X est affine. Soit U un ouvert affine dans V;  $j(X) \cap U$  est un ouvert

Figure: EGA II

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 $y \in f(X)$ . Now we can replace Y by an affine neighborhood of y, and so assume that Y is affine. Then since f is quasi-compact, X will be a finite

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## From commutative algebra to algebraic geometry

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Figure: Hartshorne, Algebraic Geometry

**Proof.** Assume X is Nagata. Let  $Z \subset X$  be an integral closed subscheme. Let  $z \in Z$ . Let  $\operatorname{Spec}(A) = U \subset X$  be an affine open containing z such that A is Nagata.

Figure: Stacks Project

## Open subsets

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• Workaround: develop the API in terms of abstract open immersions  $f: U \to X$ .

## Local properties

• Define a predicate:

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\forall (X : Scheme), P X \leftrightarrow (\forall (U : X.affineOpens), P U)
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and an induction principle:
lemma of_isLocal (P : Scheme → Prop) (h : IsLocal P) (X : Scheme)
  (hX : ∀ R, P (Spec R)) :
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• What about properties involving multiple objects and morphisms between them?

 A property of morphisms can be local at the source and at the target.

- A property of morphisms can be local at the source and at the target.
- For the target, we have:

```
class IsLocalAtTarget (P : MorphismProperty Scheme) : Prop where
iff_of_openCover :
   ∀ {X Y : Scheme} (f : X - Y) (U : Y.OpenCover),
   P f ↔ ∀ i, P (U.pullbackHom f i)
```

• This yields structured proofs:

```
lemma foo {X Y : Scheme} (f : X → Y) : P f := by
wlog hY : ∃ R, Y = Spec R
  · rw [LocalAtTarget.iff_of_openCover Y.affineCover]
  /- ... -/
obtain (R, rfl) := hY
wlog hX : ∃ S, X = Spec S
  · /- ... -/
obtain (S, rfl) := hX
obtain (φ, rfl) := Spec.map_surjective f
  /- ... -/
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- Clear separation of concerns.
- We are relying on the bundled approach here.

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#### Definition

A morphism  $f \colon X \to Y$  of schemes is *smooth* if for every  $x \in X$ , there exist affine neighbourhoods  $x \in U$  and  $f(x) \in V$  such that  $f(U) \subseteq V$  and the restriction  $f \colon U \to V$  is a smooth morphism of affine schemes.

```
/-- A morphism of schemes is smooth if locally it is a smooth morphism
of affine schemes. -/
def Smooth {X Y : Scheme} (f : X → Y) : Prop :=
    ∀ (x : X), ∃ (U : X.affineOpens) (V : Y.affineOpens),
    x ∈ U ∧ f '' U ⊆ V ∧ AffineScheme.Smooth (f.restrict U V)
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Lemma Smooth.comp {X Y Z : Scheme} {f : X → Y} {g : Y → Z}
(hf : Smooth f) (hg : Smooth g) :
Smooth (f ≫ g) :=
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lemma Smooth.comp {X Y Z : Scheme} {f : X → Y} {g : Y → Z}
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• These proofs are tedious and repetitive (we have to do this for locally of finite type, locally of finite presentation, finite, smooth, unramified, étale, flat, etc.)

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#### Proof.

The assertion is local, so it follows from the fact that the composition of smooth morphisms of affine schemes is smooth.



 From a property of morphisms on affine schemes, we obtain a property of morphisms of schemes:

```
def induced (P : MorphismProperty AffineScheme) :
    MorphismProperty Scheme :=
    fun f → ∀ (x : X), ∃ (U : X.affineOpens) (V : Y.affineOpens),
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```

• We immediately obtain a definition of smooth:

```
def Smooth {X Y : Scheme} (f : X - Y) : Prop :=
induced AffineScheme.Smooth f
```

• We can now define meta properties:

```
def MorphismProperty.StableUnderComposition
   (P: MorphismProperty C): Prop:=
   ∀ {X Y Z : C} {f : X → Y} {g : Y → Z}, P f → P g → P (f » g)
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```

• And prove meta theorems:

```
lemma stableUnderComposition_induced
{P : MorphismProperty AffineScheme}
(h : P.StableUnderComposition) :
  (induced P).StableUnderComposition :=
    /- ... -/
```

#### No, because:

• Reductions still contain a lot of boilerplate code.

```
lemma isClosedMap_iff_specializingMap (f : X → Y) [QuasiCompact f] :
    IsClosedMap f.base ↔ SpecializingMap f.base := by
 refine (fun h → h.specializingMap, fun H → ?_)
 wlog hY : \exists R, Y = Spec R

    change topologically @IsClosedMap f

    rw [IsLocalAtTarget.iff_of_openCover Y.affineCover]
   intro i
   refine this (Y.affineCover.pullbackHom f i) ?_ (_, rfl)
   exact IsLocalAtTarget.of_isPullback (.of_hasPullback _ _) H
 obtain (S, rfl) := hY
 intro 7 h7
 replace H := hZ.stableUnderSpecialization.image H
 wlog hX : \exists R, X = Spec R
  • obtain (R, g, hg) := compactSpace_iff_exists.mp (/- ... -/)
   have inst : QuasiCompact (g » f) :=
      HasAffineProperty.iff_of_isAffine.mpr (by infer_instance)
   have := this _ (g » f) (g.base -1' Z) (hZ.preimage g.continuous)
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   exact this H (_, rfl)
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 exact PrimeSpectrum.isClosed_image_of_stableUnderSpecialization
   φ.hom Z hZ H
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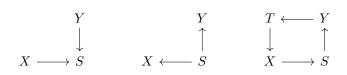
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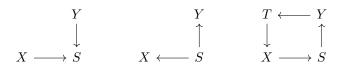
  class IsLocalAtSource (P: MorphismProperty Scheme): Prop where
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- What about diagrams with more schemes?





• Locality of properties of diagrams  $D: \mathcal{J} \to \text{Scheme}$ ?

• Encode a diagram of shape  $\mathcal J$  as a functor  $D\colon \mathcal J\to \mathrm{Scheme}.$ 

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- A localisation data of  $\mathcal{J}$  at an object  $j \in J$  is for every  $U \to D(j)$  a localised diagram  $D_U \colon \mathcal{J} \to \text{Scheme}$  with  $D_U(j) = U$ .

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- Given a localisation data of  $\mathcal{J}$  at  $j \in J$ , a property P of diagrams  $\mathcal{J} \to \text{Scheme}$  is local at j, if for every diagram  $\mathcal{J} \to \text{Scheme}$  and open cover  $(U_i)_i$  of D(j), P holds for D if and only if it holds for  $D_{U_i}$  for all i.

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- A metaprogram can construct the diagram  $\mathcal{I} \to \text{Scheme}$  from a given concrete situation and synthesize the localisation data for  $\mathcal{I}$ .

 Generalise LocalAtTarget etc., to other topologies beyond the Zariski topology (ongoing).



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