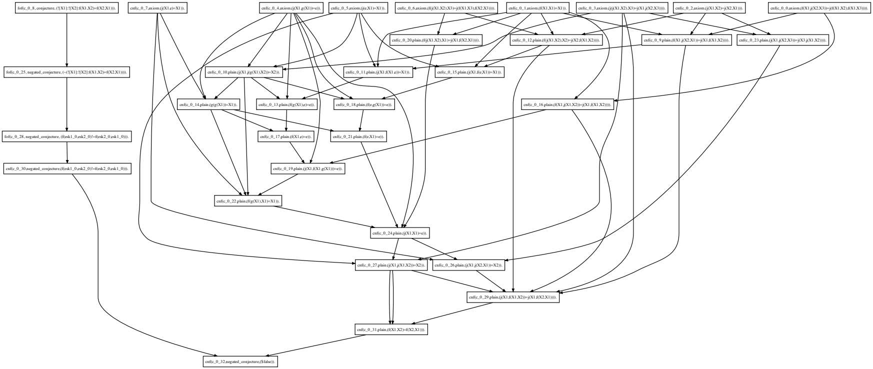

Credo quia absurdum (?)

Proof generation and challenges of proof generation

Stephan Schulz

DHBW
Duale Hochschule
Baden-Württemberg
Stuttgart

ZfKI

Agenda

1. Structure and Representation of Proofs
2. Proof Generation
3. Proof Applications
4. Challenges
5. Conclusion

Structure and Representation of Proofs

Refutational Theorem Proving

$$\{A_1, A_2, \ldots, A_n\} \models C$$

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$$\{A_1, A_2, \ldots, A_n\} \models C$$

iff

$\{A_1, A_2, \ldots, A_n, \neg C\}$ is unsatisfiable

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$\text{cnf}(\{A_1, A_2, A_n, \neg C\})$ is unsatisfiable

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iff

$$\text{cnf}(\{A_1, A_2, A_n, \neg C\}) \vdash^* \Box$$

Refutational Theorem Proving

$$\{A_1, A_2, \ldots, A_n\} \models C$$

iff

$\{A_1, A_2, \ldots, A_n, \neg C\}$ is unsatisfiable

iff

$\mathrm{cnf}(\{A_1, A_2, A_n, \neg C\})$ is unsatisfiable

Clausification

iff

$$\mathrm{cnf}(\{A_1, A_2, A_n, \neg C\}) \vdash^{*} \Box$$

Refutation/
Saturation

Ideal: Proofs as Sequences of Proof Steps

- A derivation is a list of steps
- Each step carries a clause/formula
- Each step is either. . .
    - Assumed (e.g. axioms, conjecture)
    - Logically derived from earlier steps
- A proof is a derivation that either. . .
    - derives the conjecture
    - derives a contradiction from the negated conjecture

Good mental model!

Reality: Proofs as Sequences of Proof Steps

- Initial clauses/formulas
    - Axioms/Conjectures/Hypotheses
    - Justified by assumption
- Derived clauses/formulas
    - Justified by reference to (topologically) preceding steps
    - Defined logical relationship to predecessors
        - Most frequent case: theorem of predecessors
        - Exceptions: Skolemization, negation of conjecture, . . .
- (Introduced definitions)
    - Don’t affect satisfiability/provability
    - Justified by definition

TPTP-3 language

- Consistent syntax for different classes
  - CNF is sub-case of FOF
  - FOF is sub-case of TFF
- Applicable for a wide range of applications
  - Problem specifications
  - Proofs/derivations
  - Models
- Easily parsable
  - Prolog-parsable
  - Lex/Yacc grammar
  - Recursive-descent with 1-token look-ahead
- Widely used and supported
  - CASC
  - Major provers (E, SPASS, Vampire, iProver, ...)
  - Used by integrators

Example

```
fof(c_0_0, conjecture, (?[X3]:(human(X3)&X3!=john)), file(’humen.p’, someone_not_john)).
fof(c_0_1, axiom, (?[X3]:(human(X3)&grade(X3)=a)), file(’humen.p’, someone_got_an_a)).
fof(c_0_2, axiom, (grade(john)=f), file(’humen.p’, john_failed)).
fof(c_0_3, axiom, (a!=f), file(’humen.p’, distinct_grades)).
fof(c_0_4, negated_conjecture, (˜(?[X3]:(human(X3)&X3!=john))),
    inference(assume_negation,[status(cth)],[c_0_0])).
fof(c_0_5, negated_conjecture, (![X4]:(˜human(X4)|X4=john)),
    inference(variable_rename,[status(thm)],[inference(fof_nnf,[status(thm)],[c_0_4])])).
fof(c_0_6, plain, ((human(esk1_0)&grade(esk1_0)=a)),
    inference(skolemize,[status(esa)],[inference(variable_rename,[status(thm)],[c_0_1])])).
cnf(c_0_7,negated_conjecture,(X1=john|˜human(X1)),
    inference(split_conjunct,[status(thm)],[c_0_5])).
cnf(c_0_8,plain,(human(esk1_0)),
    inference(split_conjunct,[status(thm)],[c_0_6])).
cnf(c_0_9,plain,(grade(esk1_0)=a),
    inference(split_conjunct,[status(thm)],[c_0_6])).
cnf(c_0_10,negated_conjecture,(esk1_0=john),
    inference(spm,[status(thm)],[c_0_7, c_0_8])).
cnf(c_0_11,plain,(grade(john)=f),
    inference(split_conjunct,[status(thm)],[c_0_2])).
cnf(c_0_12,plain,(a!=f),
    inference(split_conjunct,[status(thm)],[c_0_3])).
cnf(c_0_13,plain,($false),
    inference(sr,[status(thm)],[inference(rw,[status(thm)],
    [inference(rw,[status(thm)],[c_0_9, c_0_10]), c_0_11]), c_0_12]), [’proof’]).
```

```
cnf(c_0_13,
    plain,
    ($false),
    inference(sr,[status(thm)],
              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
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                          c_0_11]),
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(cnf, fof, tff, ...)

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Language
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Name
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Type
(axiom,lemma,
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```
cnf(c_0_13,
    plain,
    ($false),
    inference(sr,[status(thm)],
              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
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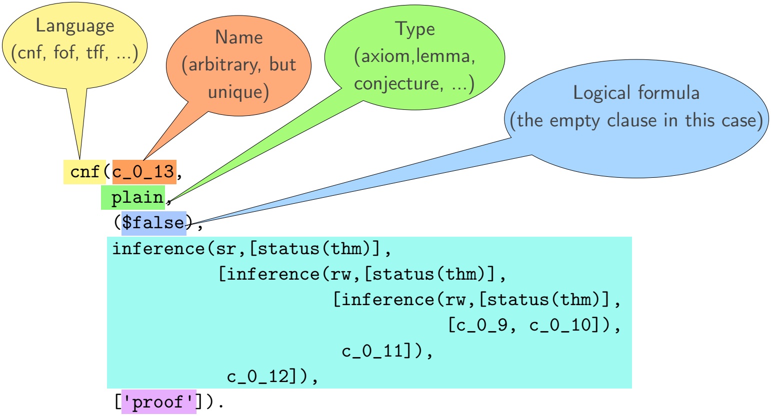
```



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Logical formula
(the empty clause in this case)

```
cnf(c_0_13,
    plain,
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    inference(sr,[status(thm)],
              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
                                    [c_0_9, c_0_10]),
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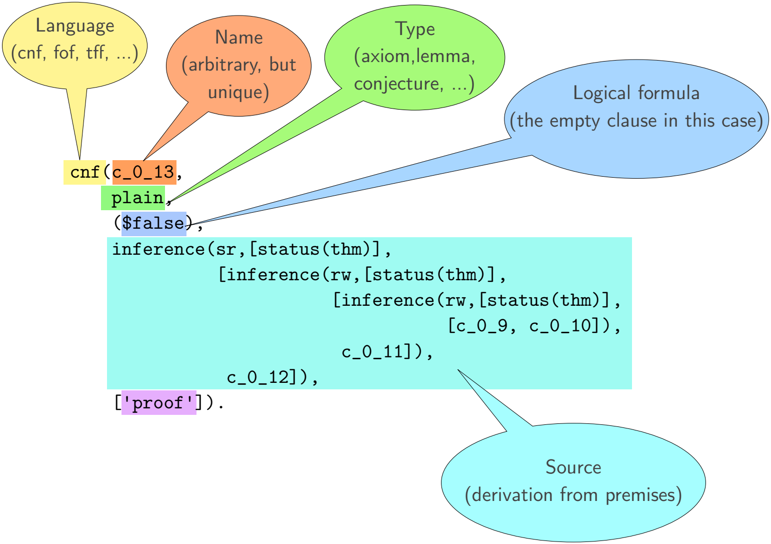

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                          c_0_11]),
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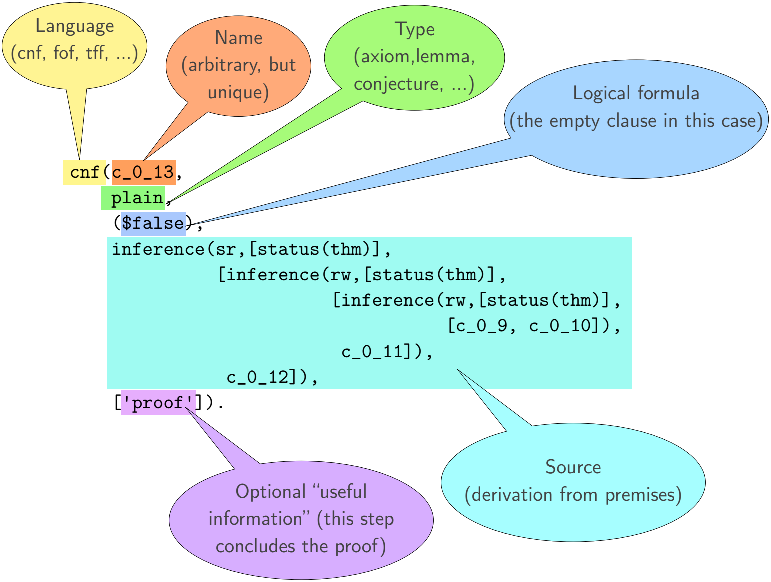
```

Source
(derivation from premises)



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                                     [c_0_9, c_0_10]),
                          c_0_11]),
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```

Source (derivation from premises)

Optional “useful information” (this step concludes the proof)

```
cnf(c_0_13,
    plain,
    ($false),
    inference(sr,[status(thm)],
              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
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cnf(c_0_13,
    plain,
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    inference(sr,[status(thm)],
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```

Inference rule (sr: Simplify-reflect, rw: Rewriting, pm: Paramodulation, ...)

```
cnf(c_0_13,
    plain,
    ($false),
    inference(sr,[status(thm)],
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Inference rule (sr: Simplify-reflect, rw: Rewriting, pm: Paramodulation, ...)

"Useful information": logical status (formula is theorem of premises)

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              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
                                    [c_0_9, c_0_10]),
                          c_0_11]),
               c_0_12]),
    ['proof']).
```

Names of the premises

```
cnf(c_0_13,
    plain,
    ($false),
    inference(sr,[status(thm)],
              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
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c_0_9:  grade(esk1_0)=a
c_0_10: esk1_0=john
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c_0_12: a!=f
```

```
cnf(c_0_13,
    plain,
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c_0_9:  grade(esk1_0)=a
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c_0_11:grade(john)=f
c_0_12: a!=f
```

Innermost inference:
Rewrite c_0_9 with
c_0_10

```
cnf(c_0_13,
    plain,
    ($false),
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```

Innermost inference: Rewrite c_0_9 with c_0_10

`grade(john)=a`

Intermediate inference: Rewrite the result of the innermost inference with c_0_11

`f=a`

```
cnf(c_0_13,
    plain,
    ($false),
    inference(sr,[status(thm)],
              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
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c_0_9:  grade(esk1_0)=a
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```

Innermost inference: Rewrite c_0_9 with c_0_10

`grade(john)=a`

Outermost (final) inference: Cut off a literal from the result of the intermediate inference with c_0_12

Intermediate inference: Rewrite the result of the innermost inference with c_0_11

```
cnf(c_0_13,
    plain,
    ($false),
    inference(sr,[status(thm)],
              [inference(rw,[status(thm)],
                         [inference(rw,[status(thm)],
                                    [c_0_9, c_0_10]),
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c_0_11:grade(john)=f
c_0_12: a!=f
```

Innermost inference: Rewrite c_0_9 with c_0_10

grade(john)=a

Compl[ie]mentary Example

```
fof(c_0_1,
    axiom,
    (?[X3]:(human(X3)&grade(X3)=a)),
    file('humen.p', someone_got_an_a)).
```

TPTP v3 idiosyncrasies

- No inference semantics
  - Rules are just names
  - Rules are system-dependent
- Incomplete inference description
  - “Rules are just names”
  - No wide support for position information

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- Workarounds:
  - Inference status
  - Proof reconstruction

Proof Generation

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$$\{A_1, A_2, \ldots, A_n\} \models C$$

iff

$\{A_1, A_2, \ldots, A_n, \neg C\}$ is unsatisfiable

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$\mathrm{cnf}(\{A_1, A_2, A_n, \neg C\})$ is unsatisfiable

Clausification

iff

$$\mathrm{cnf}(\{A_1, A_2, A_n, \neg C\}) \vdash \Box$$

Refutation

Clausification and Saturation

- Clausification
  - Terminating
  - (Usually) deterministic
  - (Usually) non-destructive
  - Sometimes done by external tool
- Saturation
  - Many degrees of freedom
  - Arbitrary search time
  - Generating inferences
    - Create new clauses
    - Necessary for completeness
  - Simplifying inferences
    - Modify/remove existing clauses
    - Necessary for performance

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- Recording clausification is straightforward
  - . . . but not always done
- Efficiently recording saturation is difficult
  - . . . some settle for inefficient

Deduction vs. Simplification

- Superposition $$\frac{s \simeq t \vee S \quad u \not\simeq v \vee R}{\sigma(u[p \leftarrow t] \not\simeq v \vee S \vee R)}$$

  if $\sigma = mgu(u|_p, s)$, [. . .]

- Rewriting $$\begin{array}{c} s \simeq t \quad u \not\simeq v \vee R \\ \hline\hline s \simeq t \quad u[p \leftarrow \sigma(t)] \not\simeq v \vee R \end{array}$$

  if $u|_p = \sigma(s)$ and $\sigma(s) > \sigma(t)$

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- Generating inferences *create* new clauses

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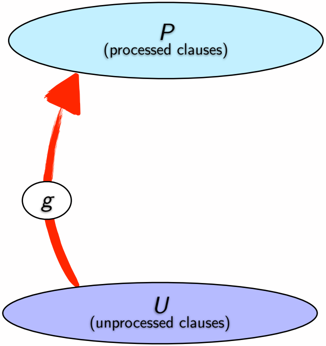
- Generating inferences *create* new clauses
- Simplifiying inferences *modify* or remove clauses

The Given-Clause Algorithm



- Aim: Move everything from *U* to *P*

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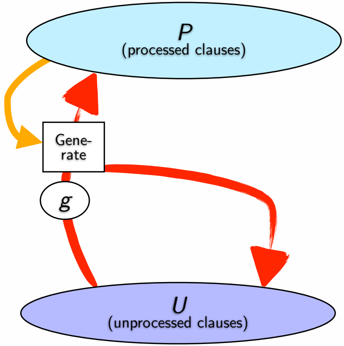

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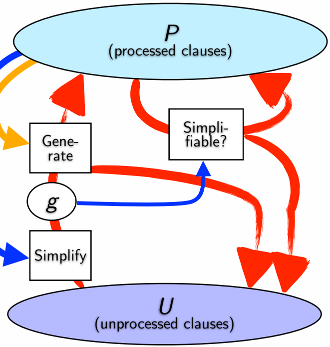

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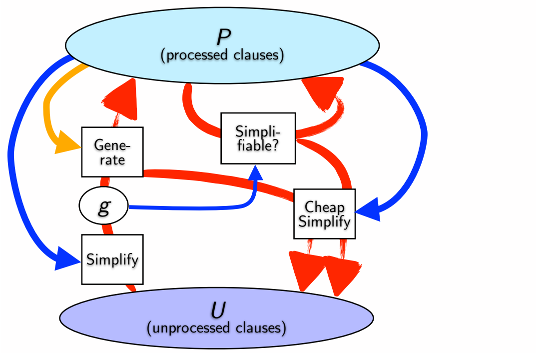

- Aim: Move everything from *U* to *P*
- Invariant: All generating inferences with premises from *P* have been performed
- Invariant: *P* is interreduced
- Clauses added to *U* are simplified with respect to *P*

Naive Proof Generation

- Basic approach:
  - Store (or dump) all intermediate proof steps
  - Extract proof steps in post-processing
- Problem: Necessary steps only known after the proof concludes
  - Intermediate results are expensive to store
  - Example: A ring with $X^4 = X$ is Abelian
    - Proof search (E): 5.4s
    - Proof search with inference dump: 11.4s
    - Post-processing: 17.6s
    - Temporary file size: 480 000 steps, 117MB
    - Proof size: 154 steps, 31 kB

Only suitable for small problems/short run-times

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  if $\sigma = mgu(u|_p, s)$, $[\dots]$

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- Generating inferences *create* new clauses
- Simplifiying inferences *modify* or remove clauses

Typical Clause Lifecycle

- Generating inference *creates* a new clause
  - Usually paramodulation (but may be equality factoring, equality resolution, . . .)
  - This also creates a new clause object
- Simplifying inferences *modify* the clause
  - Multiple rewrite steps
  - Possibly literal cutting, trivial literal removal, . . .
  - This modifies the existing clause object. . .
  - $\approx$10 modifications per clause on average (varies wildly)
- Deleting inference *removes* clause
  - Subsumption
  - Tautology deletion
  - Typically $\approx$90% of all clauses

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90% of clauses eventually deleted, 9 modified versions
$\Longrightarrow$ 99% of (logical) clauses are not persistent

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  - Tautology deletion
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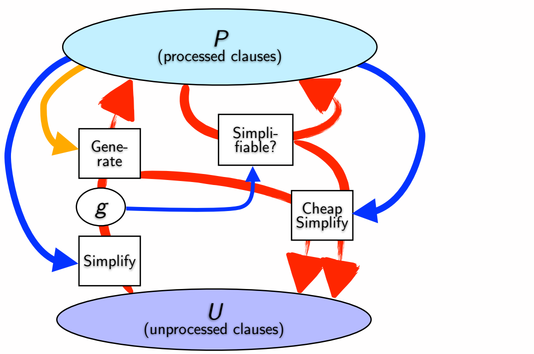
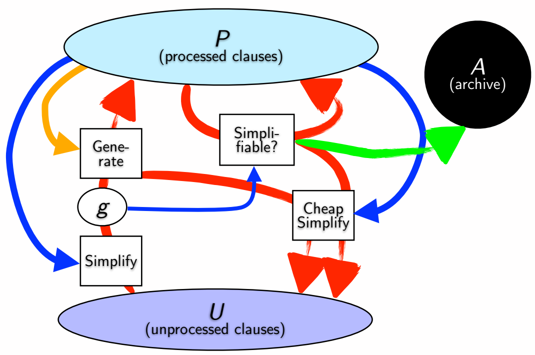
Storing all clauses is too expensive, but
we don’t know a-priori which clauses are needed!

Optimized Proof Object Construction



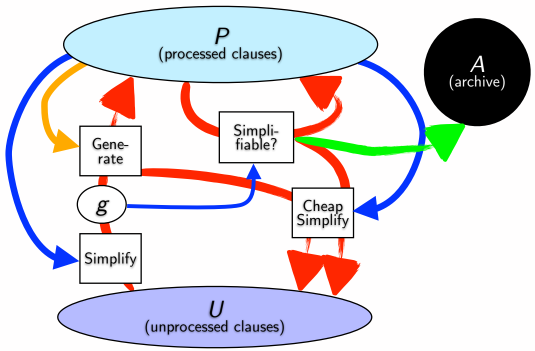
- Observation: Only clauses in *P* are premises!

Optimized Proof Object Construction



- Observation: Only clauses in *P* are premises!
- Proof recording:
  - Simplified *P*-clauses are archived
  - Clauses record their history
    - Inference rules
    - *P*-clauses involved

Optimized Proof Object Construction



- Observation: Only clauses in *P* are premises!
- Proof recording:
  - Simplified *P*-clauses are archived
  - Clauses record their history
    - Inference rules
    - *P*-clauses involved
- Proof extraction
  - Track parent relation
  - Topological sort
  - Print proof

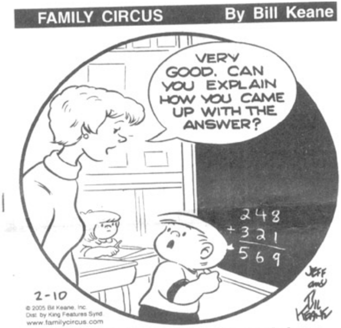
Optimized Proof Generation

- Example: A ring with $X^4 = X$ is Abelian
  - Naive approach
    - Proof search (E): 5.4s
    - Proof search with inference dump: 11.4s
    - Post-processing: 17.6s
    - Temporary file size: 480 000 steps, 117MB
    - Proof size: 154 steps, 31 kB
  - Optimized approach
    - Proof search (E): 5.5s
    - Proof search with inference dump: -
    - Post-processing: -
    - Temporary file size: -
    - Proof size: 154 steps, 31 kB
  - Example is typical
    - Optimized overhead: 0.24% over TPTP 5.4.0

Proof Applications

Why Proofs?

- Trust
  - in the proof
  - in the ATP system
  - in the specification
- Understanding
  - of the proof
  - of the domain
  - of the search process
- Learning
  - of important domain statements
  - of search control information
  - of the domain structure

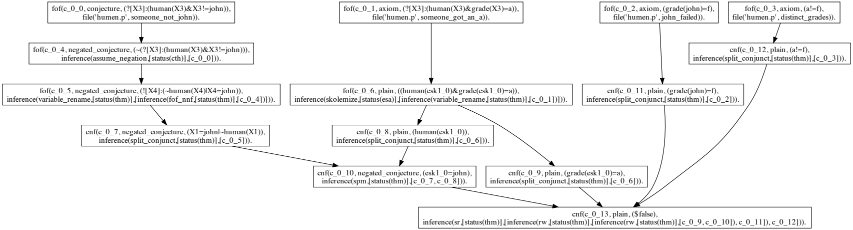


"No, I think it was just Divine intervention."

Proof Checking

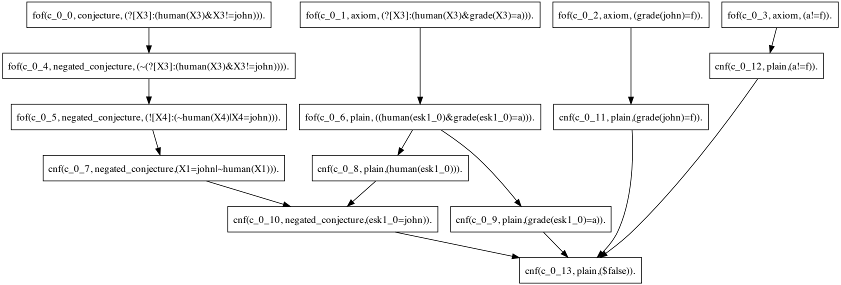
- Semantic proof checking
  - Step-by-step check
  - Verify semantic status (conclusion can be derived “somehow” from premises)
  - Use alternative theorem prover (or configuration)
- Syntactic proof checking
  - Show correctness of individual inference rule applications
  - With TPTP syntax: Requires proof reconstruction
  - E.g. Metis in Isabelle/Sledgehammer

Proof Visualization



```
fof(c_0_0, conjecture, (?[X3]:(human(X3)&X3!=john)),
file('humen.p', someone_not_john)).

fof(c_0_1, axiom, (?[X3]:(human(X3)&grade(X3)=a)),
file('humen.p', someone_got_an_a)).

fof(c_0_2, axiom, (grade(john)=f),
file('humen.p', john_failed)).

fof(c_0_3, axiom, (a!=f),
file('humen.p', distinct_grades)).

fof(c_0_4, negated_conjecture, (~(?[X3]:(human(X3)&X3!=john))),
inference(assume_negation,[status(cth)],[c_0_0])).

cnf(c_0_12, plain, (a!=f),
inference(split_conjunct,[status(thm)],[c_0_3])).

fof(c_0_5, negated_conjecture, (![X4]:(~human(X4)|X4=john)),
inference(variable_rename,[status(thm)],[inference(fof_nnf,[status(thm)],[c_0_4])])).

fof(c_0_6, plain, ((human(esk1_0)&grade(esk1_0)=a)),
inference(skolemize,[status(esa)],[inference(variable_rename,[status(thm)],[c_0_1])])).

cnf(c_0_11, plain, (grade(john)=f),
inference(split_conjunct,[status(thm)],[c_0_2])).

cnf(c_0_7, negated_conjecture, (X1=john|~human(X1)),
inference(split_conjunct,[status(thm)],[c_0_5])).

cnf(c_0_8, plain, (human(esk1_0)),
inference(split_conjunct,[status(thm)],[c_0_6])).

cnf(c_0_10, negated_conjecture, (esk1_0=john),
inference(spm,[status(thm)],[c_0_7, c_0_8])).

cnf(c_0_9, plain, (grade(esk1_0)=a),
inference(split_conjunct,[status(thm)],[c_0_6])).

cnf(c_0_13, plain, ($false),
inference(sr,[status(thm)],[inference(rw,[status(thm)],[inference(rw,[status(thm)],[c_0_9, c_0_10]), c_0_11]), c_0_12])).
```

Proof Visualization



fof(c_0_0, conjecture, (?[X3]:(human(X3)&X3!=john))).

fof(c_0_1, axiom, (?[X3]:(human(X3)&grade(X3)=a))).

fof(c_0_2, axiom, (grade(john)=f)).

fof(c_0_3, axiom, (a!=f)).

fof(c_0_4, negated_conjecture, (~(?[X3]:(human(X3)&X3!=john)))).

cnf(c_0_12, plain,(a!=f)).

fof(c_0_5, negated_conjecture, (![X4]:(~human(X4)|X4=john))).

fof(c_0_6, plain, ((human(esk1_0)&grade(esk1_0)=a))).

cnf(c_0_11, plain,(grade(john)=f)).

cnf(c_0_7, negated_conjecture,(X1=john|~human(X1))).

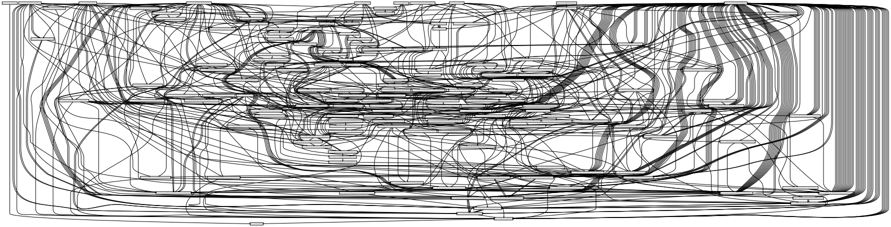
cnf(c_0_8, plain,(human(esk1_0))).

cnf(c_0_10, negated_conjecture,(esk1_0=john)).

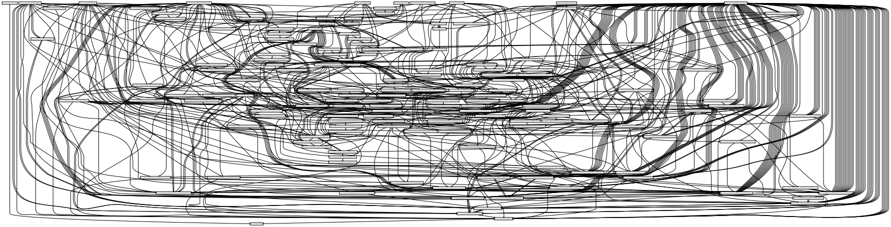
cnf(c_0_9, plain,(grade(esk1_0)=a)).

cnf(c_0_13, plain,($false)).

Another Example

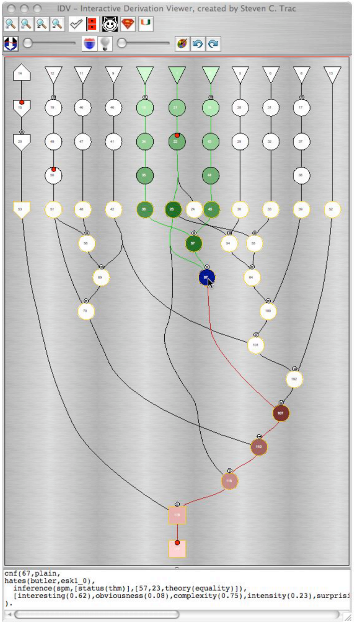


Another Example

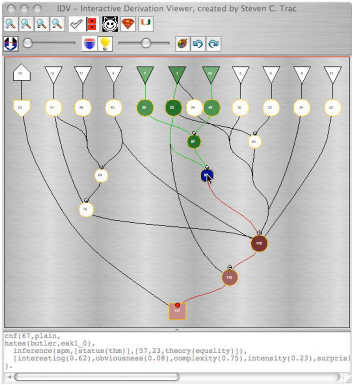


(A ring with $X^4 = X$ is Abelian)

Interactive Visualization



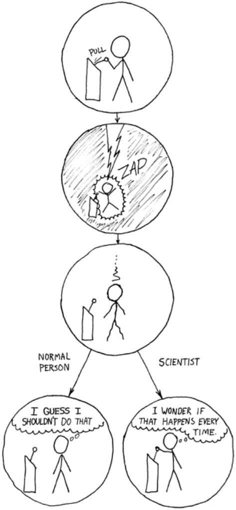
```
cnf(67,plain,
hates(butler,esk1_0),
   inference(spm,[status(thm)],[57,23,theory(equality)]),
   [interesting(0.62),obviousness(0.08),complexity(0.75),intensity(0.23),surprisi
).
```



```
cnf(67,plain,
hates(butler,esk1_0),
   inference(spm,[status(thm)],[57,23,theory(equality)]),
   [interesting(0.62),obviousness(0.08),complexity(0.75),intensity(0.23),surprisi
).
```

Learning

- Heuristics learning
  - Find formulas that frequently appear in proofs
  - Generalize and reuse
- Axiom selection
  - Learn relationship between conjecture and useful axioms
- …
- Image credit: `http://xkcd.com/242/`



Challenges

Unambiguous Inferences

- Complete inference records
  - Add inference positions
  - Add unifiers (if neccessary, e.g. HO)
  - . . .
- Complete clausification records
  - Clause simplification as rewriting (?)
  - Mini-scoping as rewriting (?)
  - Step-by-step skolemization

Theoretically managable, but practically difficult
– especially retroactively

Proof Expansion

- Calculus level expansion
  - Explicit results of each inference
  - Good for semantic proof checking
  - Good for understanding the structure of the proof
  - Potentially good for machine learning
- Primitive inferences
  - Convert inferences into primitive operations
  - For superposition:
    - Instantiation
    - Lazy conditional term replacement
    - Deleting trivial and duplicated literals
  - Uniform proof format for different provers/calculi
  - Uniform post-processing (proof checking, proof presentation, . . .)

Proof Structuring and Presentation

- Convert proof by contradiction to forward proof
  - (Jasmin Blanchette)
- Find good lemmas to structure proof
  - Syntactic features
  - Proof graph analysis
- Human-readable (?) proofs
  - Identify main lines of reasoning
  - Disentangle proof and provide focus
  - (Partially) translate to natural language

Conclusion

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- TPTP v3 is a useful and used standard for proof representation
- Proof objects are useful for trust building and learning
- Use of proof objects is still in its infancy - we need more tools

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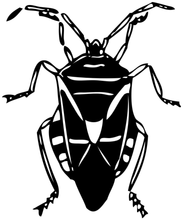
Proof presentation is a big open area

Ceterum Censeo. . .



- Bug reports for E should include:
  - The exact command line leading to the bug
  - All input files needed to reproduce the bug
  - A description of what seems wrong
  - **The output of `eprover --version`**