Linear Types with Dynamic Multiplicities in Dependent Type Theory

WG6 meeting, Genoa

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Specifying variable use with linear types

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called *multiplicity* of x

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Proposal: impose linear rules *inside* dependent type theory.
This allows us to have *dynamic/dependent* multiplicities.

$$\Gamma \vdash \underbrace{\Delta \Vdash A}_{\text{defined as certain dependent type}}$$

(Linear) judgment day

Cubical Agda supports finite multisets, which behave just like lists except that the order of elements does not matter.

```
data FMSet (A : Type) : Type where
  ◇     : FMSet A
  _::_   : A → FMSet A → FMSet A
  comm  : ∀ x y xs → x :: y :: xs ≡ y :: x :: xs
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we already have many useful equalities, e.g., `swap : Δ₀ ⊗ Δ₁ ≡ Δ₁ ⊗ Δ₀`

Same, but different. But still same

```
switch : (z : A × B) → η z ⊩ B × A
switch (x , y) = (y , x) ,{Goal: η (x , y) “≡“ η (y , x) }
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Adding and removing pair constructor doesn’t change the free variables of a supply.
→ introduce notion of sameness for supplies, which we call *productions*.

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    cn, : η (a , b) ⋈ (η a ⊗ η b) : wk,        (for a : A, b : B a)
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A natural resource algebra

We get multiplicities for free using the standard natural numbers type:

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_^_ : Supply → ℕ → Supply
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```
copytwice : (x : A) → η x ^ 4 ⊩ (A × A) × (A × A)
copytwice = compose copy copy
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…but this directly computes!

Programming linearly with lists

We introduce productions for the lists constructors:

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```
((x : A) → η x ⊗ Δ₀ ⊩ B) → Δ₁ ⊩ A → Δ₀ ⊗ Δ₁ ⊩ B
```

Recap

- Supplies as *finite multisets of pointed types* are a useful notion of resource, dependent pairs allow us to define a linear judgment *inside* type theory.

$$\boxed{\Delta \Vdash A = \Sigma[\ a \in A\ ]\ (\Delta \bowtie \eta\ a)}$$

- Productions capture which supplies have the *same multiset of free variables*. Incorporate datatypes by stipulating productions for each constructor.
  → quantitative elimination principles are *derived* using dependent elimination!

- Dependent types are naturally part of the system.

- This is already practical for programming, for example it’s easy to construct sorting algorithms. Simple tactic could automatically find most productions.

Leaving cubical behind

We can carry out our construction in any dependent type theory with Π and Σ:

$$\begin{tikzcd} Tm \arrow[d, "\pi"] & : Cx^{op} \to \mathbf{Set} \\ Ty & : Cx^{op} \to \mathbf{Set} \end{tikzcd}$$

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- $\eta(a) \otimes \eta(b) \simeq \eta(a, b)$ for any $a : A$ and $b : B(a)$

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$$\Gamma \vdash \Delta \Vdash A$$

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Can we internalise this structure? In other words, how to add function types?

Proposal for internalising $\Gamma \vdash \Delta \Vdash A$

Add two more things:

- exponentials $[\Delta_0, \Delta_1]$
- $\Lambda_{x:A}\Delta$ binding x in Δ

$Sp : Cx^{op} \to \mathbf{SM\underline{C}Cat}$

functor $\Lambda_A : Sp(\Gamma\,.\,A) \to Sp(\Gamma)$ that's right adjoint to context extension $Sp(\mathbf{p}_A) : Sp(\Gamma) \to Sp(\Gamma\,.\,A)$

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This allows us to define a type of dependent linear functions from A to B:

$$(x : A) \multimap B(x) := (x : A) \to B(x)\ ,\ \lambda f \to \Lambda_{x:A}[\eta(x), \eta(f\ x)]$$

We can derive intuitive introduction and elimination rules for $(x : A) \multimap B(x)$.

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(generalise η to dependent supplies for higher-order functions)

We can derive intuitive introduction and elimination rules for $(x : A) \multimap B(x)$.

Summary

- Adding symmetric monoidal structure to dependent type theory is useful.
  - This also happens with non-idempotent intersection types (De Carvalho, Ronchi della Rocca, Gardner), but more powerful base theory makes our life easier.
- Quantitative features come for free, multiplicities are (open) terms of type $\mathbb{N}$.
  - We can type many more programs than systems with static resource algebra (QTT, Graded TT, Linear Haskell). Observation due to Pierre-Marie Pedrót (*Dialectica the Ultimate*, talk at TLLA 2024).
- WIP: expand idea to incorporate *dependent linear function types*.
  Gives rise to a *dependent linear type theory* with *dependent multiplicities*.

https://github.com/maxdore/dltt/

Dependent linear functions

$$(x : A) \multimap B(x) := (x : A) \to B(x) \;,\; \lambda f \to \Lambda_{x:A}[\eta(x), \eta(f\ x)]$$

$$\frac{\Gamma, x : A \vdash \Delta \otimes \eta(x)^m \Vdash b : B(x)}{\Gamma \vdash \Delta \Vdash \lambda x\,.\, b : (x : A) \multimap^m B(x)} \multimap I \;(x \notin \Delta)$$

$$\frac{\Gamma \vdash \Delta_0 \Vdash f : (x : A) \multimap^m B(x) \qquad \Gamma \vdash \Delta_1 \Vdash a : A}{\Gamma \vdash \Delta_0 \otimes \Delta_1^m \Vdash f\ a : B(a)} \multimap E$$

Dependent linear type theory

We can define a type theory with linear dependent types using the following:

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$$\begin{tikzcd} Sp(\Gamma) \arrow[rr, bend left, "Sp(\mathbf{p}_A)"] \arrow[rr, phantom, "\perp"] & & Sp(\Gamma\,.\,A) \arrow[ll, bend left, "\Lambda_A"] \end{tikzcd}$$

+ for Σ types: iso between $\eta(a) \otimes \eta(b)$ and $\eta(a, b)$ for any $a : A$ and $b : B(a)$

Linear types without finite multisets

```
data Supply : Type where
  ◇ : Supply
  η : {A : Type} (a : A) → Supply
  _⊗_ : Supply → Supply → Supply

data _⋈_ : Supply → Supply → Type where
  id : ∀ Δ → Δ ⋈ Δ
  _∘_ : ∀ {Δ₀ Δ₁ Δ₂} → Δ₁ ⋈ Δ₂ → Δ₀ ⋈ Δ₁ → Δ₀ ⋈ Δ₂
  _⊗ᶠ_ : ∀ {Δ₀ Δ₁ Δ₂ Δ₃} → Δ₀ ⋈ Δ₁ → Δ₂ ⋈ Δ₃ → Δ₀ ⊗ Δ₂ ⋈ Δ₁ ⊗ Δ₃
  unitr : ∀ Δ → Δ ⊗ ◇ ⋈ Δ
  unitr' : ∀ Δ → Δ ⋈ Δ ⊗ ◇
  swap : ∀ Δ₀ Δ₁ → Δ₀ ⊗ Δ₁ ⋈ Δ₁ ⊗ Δ₀
  assoc : ∀ Δ₀ Δ₁ Δ₂ → (Δ₀ ⊗ Δ₁) ⊗ Δ₂ ⋈ Δ₀ ⊗ (Δ₁ ⊗ Δ₂)
```

Currying example

$$
\dfrac{\dfrac{\dfrac{\dfrac{x : A, y : B(x) \vdash \Delta \Vdash f : \mathring{\Pi}^{1}_{\mathsf{pair}(x,y) : \Sigma_A(B)}(C(y)) \qquad \dfrac{}{x : A, y : B(x) \vdash \eta(\mathsf{pair}(x,y)) \Vdash \mathsf{pair}(x,y) : \Sigma_A(B)}\,\textsc{Id}}{x : A, y : B(x) \vdash \Delta \otimes \eta(\mathsf{pair}(x,y)) \Vdash f(\mathsf{pair}(x,y)) : C(y)}\,\mathring{\Pi}\textsc{App}}{x : A, y : B(x) \vdash \Delta \otimes \eta(x) \otimes \eta(y) \Vdash f(\mathsf{pair}(x,y)) : C(y)}\,\omega_{\mathsf{pair}}}{x : A \vdash \Delta \otimes \eta(x) \Vdash \lambda y. f(\mathsf{pair}(x,y)) : \mathring{\Pi}^{1}_{B(x)}(C)}\,\mathring{\Pi}\mathrm{I}}{\vdash \Delta \Vdash \lambda x.\lambda y. f(\mathsf{pair}(x,y)) : \mathring{\Pi}^{1}_{x:A}(\mathring{\Pi}^{1}_{B(x)}(C))}\,\mathring{\Pi}\mathrm{I}
$$