# A Mathematical Theory of Term Relations

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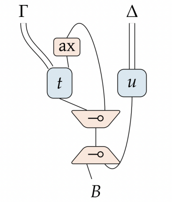
# Syntactic and Operational Semantics

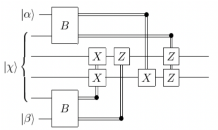
`LOAD r1,b; RET`

`x += x; return x`

$\Lambda\alpha.\lambda(x : \alpha).x : \forall\alpha.\alpha \longrightarrow \alpha$

$\Pi(x : A).\ \mathtt{refl}(x) : x =_A x$



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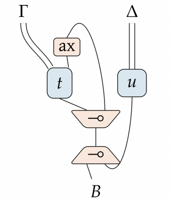


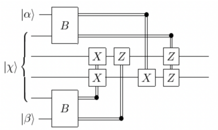

**Operational** = Meaning through **computation**

**Syntactic** = Definitions based on **syntax** (no denotations, no machines)

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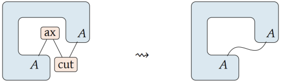


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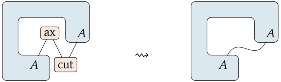


$Y \simeq \boldsymbol{\Theta}$

**Key semantic behaviours** (**Deep Invariants of Syntax**)

| | | |
|---|---|---|
| Type safety | Consistency | Congruence |
| Cut Elimination | (Confluence + Termination) | Parametricity |

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Mainstream approach: **Proof-Theoretic Semantics**

- (G)SOS: (Categorical) **Structural Rules**
- Reduction Semantics: (Categorical) **Rewriting**

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**Today** $\to$ **Relational Foundation**

# Relational Foundation: Why?

**1.** Semantics given as **relations on syntactic terms** → **Term Relations**

$$\mathtt{t} \rightarrow \mathtt{s} \quad \mathtt{t} \Downarrow \mathtt{s} \quad \Gamma \vdash \mathtt{t} \rhd \mathtt{T} \quad \Gamma \vdash \mathtt{T} \quad \mathtt{t} \simeq \mathtt{s} \quad \mathtt{t} \gtrapprox \mathtt{s} \quad \mathtt{t} =_{\varepsilon} \mathtt{s}$$

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**2.** **Semantic meta-theory** as properties of term relations

**Type Safety** $\dfrac{\mathtt{t} \rhd \mathtt{T} \quad \mathtt{t} \to \mathtt{s}}{\mathtt{s} \rhd \mathtt{T}}$ $\quad \leftarrow ; \rhd \subseteq \rhd$

**Congruence** $\dfrac{\mathtt{t}_1 \simeq \mathtt{s}_1, \ldots, \mathtt{t}_n \simeq \mathtt{s}_n}{\mathcal{C}[\mathtt{t}_1, \ldots, \mathtt{t}_n] \simeq \mathcal{C}[\mathtt{s}_1, \ldots, \mathtt{s}_n]}$ $\quad \widetilde{\simeq} \subseteq \simeq$

**Confluence** $\dfrac{\mathtt{t}_1 \leftarrow \mathtt{t} \to \mathtt{t}_2}{\exists \mathtt{s}.\ \mathtt{t}_1 \to \mathtt{s} \leftarrow \mathtt{t}_2}$ $\quad \leftarrow ; \to \subseteq \to ; \leftarrow$

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3. **Relational Proof Techniques**

- **Tait-Martin-Löf** (confluence)
- **Howe's method** (congruence)
- **Logical Relations** (parametricity)

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**A.** Term relations are **language-dependent** and **syntax-dependent**

- Relations in context, Type-respecting relations, ...

Based on syntax-specific operations

- Pattern-matching
- Unification and substitution

Built proof-theoretically ☞Operational semantics as proof-theoretic semantics

- **Induction**, **finitary rules** (type elaboration, reduction)
- **Coinduction**, **inifinitary rules** (bisimilarity, $\omega$-reduction)
- **Impredicativity**, **HO quantification** (contextual equivalence, Leibniz equality)

**NB.** $\lambda$-calculus example of all of that!

No **general** and **syntax-independent** notion of **term relation**

Term relations are useful **language-specific** **technique** rather than an independent **mathematical object**

**This Talk**:

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- Outline of a **relational foundation** of **Operational Semantics**

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**Methodology**

**Structural Analysis** ⇄ **Algebraic Synthesis**

Relational

~~Algebraic~~ Theory of Syntax

# Step 1: Relational Syntax

**Term Relations** = **Syntactically**-defined Relations on **Syntactic Terms**

**Term Relations** = **Structurally** -defined Relations on **Syntactic Strcutures**

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| **Universe of Terms** | Topos $\mathcal{E}$ $(\boldsymbol{Set}, \boldsymbol{Set}^{\boldsymbol{Ctx}}, \boldsymbol{Nom})$ |
| **Atomic Terms** | $\mathcal{E}$-object $V$ |
| **Term Constructs** | Signature Functor $\Sigma : \mathcal{E} \to \mathcal{E}$ |
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**Term Interpretation** $Hom_{\mathcal{E}}(\Sigma^\dagger V, -)$ $\to$ **Term Relation** $Rel_{\mathcal{E}}(\Sigma^\dagger V, \Sigma^\dagger V)$

# Relational Extension

Topos $\mathcal{E}$ $\longrightarrow$ **Allegory** $Rel(\mathcal{E})$

$$(\Delta, a \,;\, b, a^\circ, \bigcup_i a_i, \bigcap_i a_i)$$

Functor $\Sigma : \mathcal{E} \to \mathcal{E}$ $\longrightarrow$ **Relator** $\overline{\Sigma} : Rel(\mathcal{E}) \to Rel(\mathcal{E})$

Initial $F$-algebra $\longrightarrow$ **Relational Initial $\overline{F}$-algebra**

$$\begin{array}{ccc} V + \Sigma(\Sigma^\dagger V) & \xrightarrow{(\overline{V+\Sigma})\,(\!|a|\!)} & V + \Sigma(A) \\ {\scriptstyle [\eta,\sigma]}\downarrow & & \downarrow{\scriptstyle a} \\ \Sigma^\dagger V & \xrightarrow[(\!|a|\!)]{} & A \end{array}$$

Monad $(\mathcal{R}, \eta, \rho)$ $\longrightarrow$ **Lax monad** $(\overline{\mathcal{R}}, \eta, \rho)$

$$\begin{array}{ccc} \Sigma^\dagger\Sigma^\dagger V & \xrightarrow{\rho} & \Sigma^\dagger V \\ {\scriptstyle \overline{\Sigma^\dagger\Sigma^\dagger} a}\downarrow & \subseteq & \downarrow{\scriptstyle \overline{\Sigma^\dagger} a} \\ \Sigma^\dagger\Sigma^\dagger V & \xrightarrow[\rho]{} & \Sigma^\dagger V \end{array}$$

| **Algebraic Syntax**<br>**Topos** $\mathcal{E}$ | **Relational Syntax**<br>**Allegory** $\mathcal{A}$ |
|---|---|
| $\mathcal{E}$-object | $\mathcal{A}$-object |
| Functor $\Sigma : \mathcal{E} \to \mathcal{E}$ | Relator $\Gamma : \mathcal{A} \to \mathcal{A}$ |

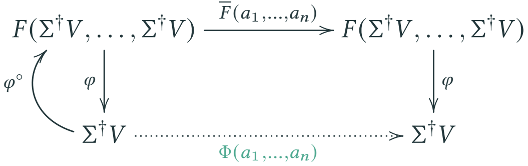
**NB.** Algebraic and relational views are **equivalent**

$$(\mathcal{E}, F) \longrightarrow (Rel(\mathcal{E}), \overline{F})$$

$$(Map(\mathcal{A}), U\Gamma) \longleftarrow (\mathcal{A}, \Gamma)$$

# Step 2: Structure of Term Relations

$$\frac{\textbf{Structure} \quad \varphi : F(\Sigma^\dagger V, \ldots, \Sigma^\dagger V) \to \Sigma^\dagger V \text{ in } \mathcal{E}}{\textbf{Operation} \quad \Phi : \mathit{Rel}(\mathcal{E})(\Sigma^\dagger V, \Sigma^\dagger V)^n \to \mathit{Rel}(\mathcal{E})(\Sigma^\dagger V, \Sigma^\dagger V)}$$




$$\Phi(\vec{a}) = \varphi^\circ ; \overline{F}(\vec{a}) ; \varphi$$

- ☞ $\Phi(a_1, \ldots, a_n)$ pattern-matches $F$-terms in $\Sigma^\dagger V$
- ☞ **McCarthy's Analytic Syntax**

How many structures for syntax?

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| **Atoms** | $\eta : V \to \Sigma^{\dagger} V$ |
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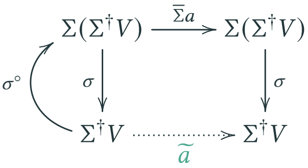
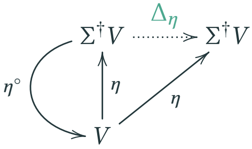
## Compatible Refinement




$$\frac{X \vdash \mathtt{t}_1 \; a \; \mathtt{s}_1 \quad \cdots \quad X \vdash \mathtt{t}_n \; a \; \mathtt{s}_n}{X \vdash \mathtt{op}(\mathtt{t}_1, \ldots, \mathtt{t}_n) \; \widetilde{a} \; \mathtt{op}(\mathtt{s}_1, \ldots, \mathtt{s}_n)}$$

## Atom Co-Reflexive




$$\frac{\mathtt{x} \in V}{X, \mathtt{x} \vdash \mathtt{x}\ \Delta_\eta\ \mathtt{x}}$$

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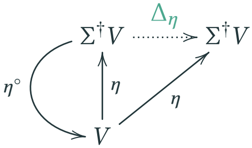


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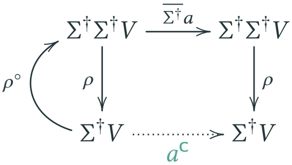
## Context




$$\frac{X \vdash \mathtt{t}_1 \ a \ \mathtt{s}_1 \quad \cdots \quad X \vdash \mathtt{t}_n \ a \ \mathtt{s}_n}{X \vdash \mathtt{C}[\mathtt{t}_1, \ldots, \mathtt{t}_n] \ a^{\mathsf{c}} \ \mathtt{C}[\mathtt{s}_1, \ldots, \mathtt{s}_n]}$$

Basic account of relational use of syntax

- ☞ A piece of syntax is (`variable` | `operator` | $\cdots$),
  if I use it as (`variable` | `operator` | $\cdots$)
- ☞ Relations as actions (Pratt) as manipulations

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**Algebraic Syntax**: $\Delta = \Delta_\eta \cup \widetilde{\Delta}$

**Structural Induction**: $\Delta_\eta \cup \widetilde{a} \subseteq a \implies \Delta_\eta \subseteq a$

**Congruence**: $\widetilde{a} \subseteq a$

**Free Monad**: $a^{\mathsf{c}} = \mu x.\Delta_\eta \cup a \cup \widetilde{x}$

$$
\begin{array}{ccc}
V + \Sigma(\Sigma^{\dagger} V) & \xrightarrow{(\overline{V+\Sigma})\,(\![\eta,\sigma];a]\!)} & V + \Sigma(\Sigma^{\dagger} V) \\
\downarrow{\scriptstyle [\eta,\sigma]} & & \downarrow{\scriptstyle [\eta,\sigma];a} \\
\Sigma^{\dagger} V & \xrightarrow{\phantom{xxxxxxxxxxxx}} & \Sigma^{\dagger} V \\
 & \underbrace{(\![\eta,\sigma];a]\!)}_{a^{\mathsf{H}}} &
\end{array}
$$

$$
\frac{}{\mathtt{x}\ a^{\mathsf{H}}\ \mathtt{x}}
\qquad
\frac{\mathtt{x}\ a\ \mathtt{t}}{\mathtt{x}\ a^{\mathsf{H}}\ \mathtt{t}}
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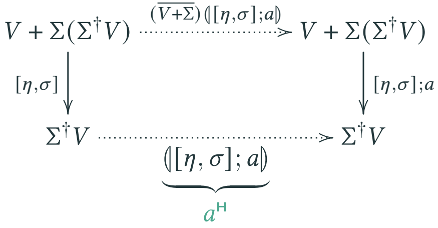


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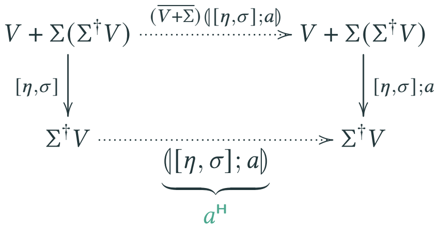
☞ $a^{\mathsf{H}}$ is the **Howe Extension** of $a$ → **Congruence of Bisimilarity**

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- ☞ $a^{\mathsf{H}}$ is the **Howe Extension** of $a$ → **Congruence of Bisimilarity**
- ☞ $(\to \cup \Delta)^{\mathsf{H}}$ is **Parallel Reduction** → **Tait-Martin-Löf Technique**

$$a[\Delta]^\circ; a[\Delta] \subseteq \Delta \;\&\; a[\Delta]^\circ; \widetilde{a^{\mathsf{H}}} \subseteq a^\circ[a^{\mathsf{H}}] \implies a^{\mathsf{H}\circ}; a^{\mathsf{H}} \subseteq a^{\mathsf{H}}; a^{\mathsf{H}\circ}$$

# Step 3: Calculus of Term Relations

Collection of operations on $Rel(\mathcal{E})(\Sigma^\dagger V, \Sigma^\dagger V)$

$$\underbrace{(\Delta, ;, \cdot^\circ, \bigcup, \bigcap)}_{\textbf{pure relational operations}} \quad + \quad \underbrace{(\Delta_\eta, \widetilde{\cdot}, \cdot^{\mathsf{H}}, \cdot^{\mathsf{C}})}_{\textbf{term relation operations}}$$

Subject to simple **calculational laws**

$$\Delta_\eta \subseteq \Delta$$

$$\widetilde{a}; \widetilde{b} = \widetilde{a; b}$$

$$\widetilde{a}^\circ = \widetilde{a^\circ}$$

$$\Delta_\eta; \widetilde{a} = \bot$$

$$\widetilde{a}^* \subseteq \widetilde{a \cup \Delta}^*$$

$$a \subseteq b \implies \widetilde{a} \subseteq \widetilde{b}$$

$$\Delta^{\mathsf{H}} = \Delta$$

$$a; (\Delta_\eta \cup \widetilde{b}) \subseteq b \implies a^{\mathsf{H}} \subseteq b$$

$$\Delta_\eta \cup a \cup \widetilde{b} \subseteq b \implies a^{\mathsf{C}} \subseteq b$$

# Step 4: Term Relation Algebras

**Calculus of Term Relations** internal to $Rel(\mathcal{E})(\Sigma^{\dagger}V, \Sigma^{\dagger}V)$

- ☞ **Building blocks** and **reasoning principles** of term relations
- ☞ Isolates the very **structure** of term relations

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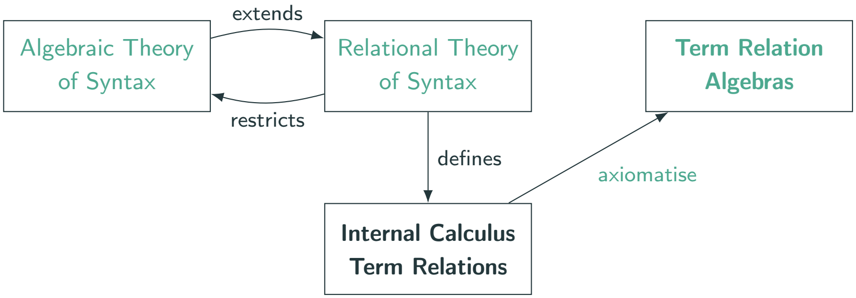

# Term Relation Algebras

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- ☞ Just one kind of object: **relations**
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TRAs are **easy to use**

- ☞ Directly endow conrete formalisms with TRA structure

$$\frac{\Gamma, \mathtt{x} : \mathtt{T} \vdash \mathtt{t} \; a \; \mathtt{s} : \mathtt{S}}{\Gamma \vdash \lambda(\mathtt{x} : \mathtt{T}).\mathtt{t} \; \widetilde{a} \; \lambda(\mathtt{x} : \mathtt{T}).\mathtt{s} : \mathtt{T} \to \mathtt{S}} \qquad \frac{\Gamma, \mathtt{a} \vdash \mathtt{t} \; a \; \mathtt{s} : \mathtt{T}}{\Gamma \vdash \Lambda \mathtt{a}.\mathtt{t} \; \widetilde{a} \; \Lambda \mathtt{a}.\mathtt{s} : \forall \mathtt{a}.\mathtt{T}}$$

$$\frac{\Gamma \vdash \mathtt{t} \; a \; \mathtt{t}' : \mathtt{S} \to \mathtt{T} \quad \Gamma \vdash \mathtt{s} \; a \; \mathtt{s}' : \mathtt{S}}{\Gamma \vdash \mathtt{ts} \; \widetilde{a} \; \mathtt{t}'\mathtt{s}' : \mathtt{T}}$$

- ☞ Quasi-equational reasoning → **mechanised semantics**

TRAs capture **operational use** of syntax

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| **Atoms** | $\eta : V \to \Sigma^{\dagger} V$ |
| **Term Constructs** | $\sigma : \Sigma(\Sigma^{\dagger} V) \to \Sigma^{\dagger} V$ |
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No matter which structure used for substitution: $a[b] = \zeta^\circ.\ (a \mathbin{\overline{\otimes}} b); \zeta$

$$(a;c)[b;d] \subseteq a[b];c[d] \qquad (a[b])[c] = a[b[c]] \qquad \widetilde{a}[b] \subseteq \widetilde{a[b]}$$

$$a[b]^\circ = a^\circ[b^\circ] \qquad \Delta_\eta[a] = a = a[\Delta_\eta] \qquad a[b] \subseteq c \iff a \subseteq b \gg c$$

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No matter which structure used for substitution: $a[b] = \zeta^\circ.\ (a \,\overline{\otimes}\, b); \zeta$

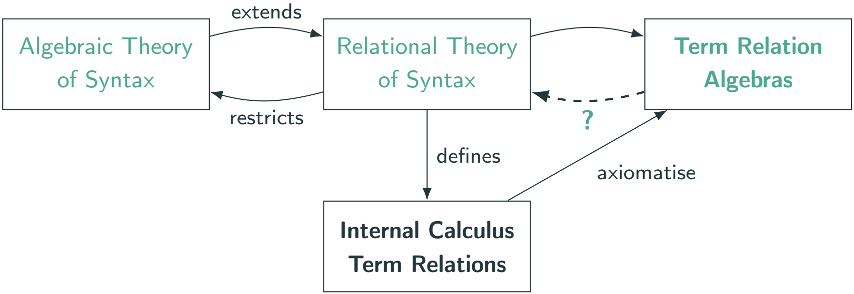
$$(a;c)[b;d] \subseteq a[b];c[d] \quad (a[b])[c] = a[b[c]] \quad \widetilde{a}[b] \subseteq \widetilde{a[b]}$$

$$a[b]^\circ = a^\circ[b^\circ] \quad \Delta_\eta[a] = a = a[\Delta_\eta] \quad a[b] \subseteq c \iff a \subseteq b \gg c$$

$$\mathtt{x}_1, \ldots, \mathtt{x}_n \vdash \mathtt{t} \ \ a \gg b \ \ \mathtt{s} \iff \forall \tau, \sigma.\ Y \vdash \tau(\mathtt{x}_i) \ \ a \ \ \sigma(\mathtt{x}_i) \implies Y \vdash \mathtt{t}[\tau] \ \ b \ \ \mathtt{s}[\sigma]$$

☞ Logical Relations

# Relational Syntax, Revisited



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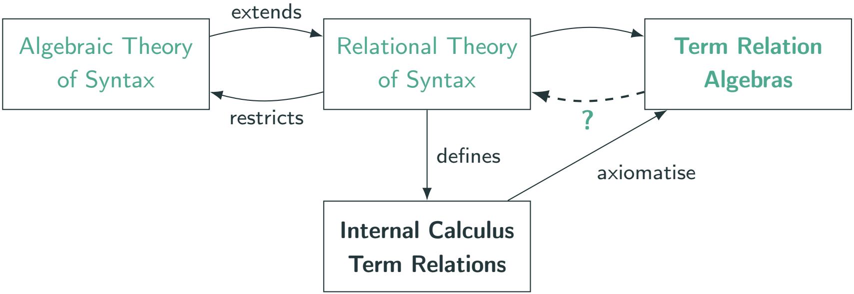
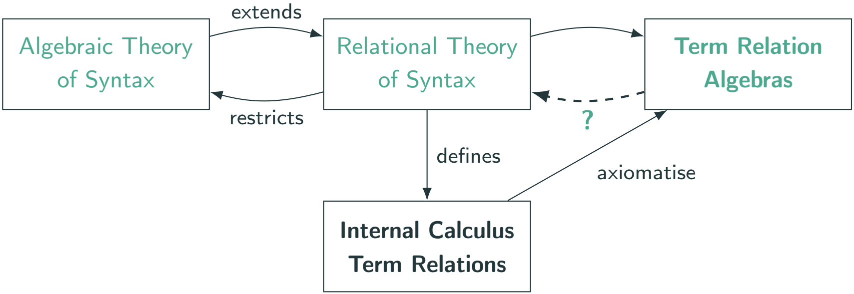



| **Structure** | $\varphi : F(\Sigma^{\dagger}V, \ldots, \Sigma^{\dagger}V) \to \Sigma^{\dagger}V$ in $\mathcal{E}$ |
|---|---|
| **Operation** | $\Phi : Rel(\mathcal{E})(\Sigma^{\dagger}V, \Sigma^{\dagger}V)^{n} \to Rel(\mathcal{E})(\Sigma^{\dagger}V, \Sigma^{\dagger}V)$ |

**Q.** How much information do we loose?

# Relational Syntax, Revisited




$$\frac{\textbf{Structure} \quad \varphi : F(\Sigma^\dagger V, \ldots, \Sigma^\dagger V) \to \Sigma^\dagger V \text{ in } \mathcal{E}}{\textbf{Operation} \quad \Phi : Rel(\mathcal{E})(\Sigma^\dagger V, \Sigma^\dagger V)^n \to Rel(\mathcal{E})(\Sigma^\dagger V, \Sigma^\dagger V)}$$

**Q.** How much information do we loose?

**A.** No relevant information is lost

***Representation Theorem.*** Any TRA arises as the algebra of term relations of a syntactic structure.

## Expressiveness

# How Far Can We Go?

**Q**. How much operational semantics can we within TRAs?

| | | |
|---|---|---|
| $\alpha$**TRA** | | **Atoms** |
| | | **Operators** |
| | | **Contexts** |
| $\sigma$**TRA** | **Substitution** | |
| $\partial$**TRA** | **Linear Contexts** | |

Enough for **rewriting**, **congruence**, **equational deduction**

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Enough for **rewriting**, **congruence**, **equational deduction**

What about

> **program statics** and **program dynamics**?
>
> **Behavioural equivalences**
>
> **Computational interpretation of proofs**?

# A Computational Pre-Theory of Syntax

Martin-Löf (Constructive Mathematics and Computer Programming)

| | | |
|---|---|---|
| **Operators** | **Intro Forms**<br>**Elim Forms** { **majo arg.** / **minor arg.** | $\mathsf{abs}(\mathtt{x}.-)$<br>$\mathsf{app_n}(\bullet, \circ)$<br>$\mathsf{app_v}(\bullet, \bullet)$ |
| **Terms** | **Complete**<br>**Incomplete** | $\emptyset \vdash \mathsf{lam}(\mathtt{x}.\mathtt{x})$<br>$\mathtt{x} \vdash \mathsf{app_n}(\mathtt{x}, \mathtt{x})$ |
| **Computation** | **Gentzen's Principle** | $\mathsf{elim}(\underbrace{\mathsf{intro}}_{\text{major}}, -)$ |

| | | |
|---|---|---|
| **Intro Forms**<br>**Elim Forms** | $I : \mathcal{E} \to \mathcal{E}$<br>$E : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ | $\overline{a}$<br>$\langle a, b \rangle$ |
| **Complete**<br>**Incomplete** | $\mathcal{E}_0 \underset{U}{\overset{\Delta}{\rightleftarrows}} \mathcal{E} \quad \Delta \dashv U$ | $\Box \dashv \blacklozenge$ |
| **Gentzen's Principle** | | $a \subseteq \langle \overline{\Delta}, \Delta \rangle; a$ |

| | | |
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## $\varphi$**TRA**

About 25 axioms (but they are decreasing)

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Subsumes other TRAs (e.g. $\widetilde{a} = \overline{a} \cup \langle a, a \rangle$)

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## $\varphi$TRA

About 25 axioms (but they are decreasing)

Subsumes other TRAs (e.g. $\widetilde{a} = \overline{a} \cup \langle a, a \rangle$)

Many examples

- **CbN**, **CbV**, **FG-CbV**
- **System T**
- **System F$\mu$**
- $\lambda\Pi$
- **Continuations**
- **Object Calculi** ($\varsigma$-calculus)
- **Natural Deduction**

# Expressiveness

**Dynamic Semanrics**

$$a^{\mathsf{E}} = \mu x.\overline{\Delta} \cup \langle x, \Delta \rangle; a; x$$

**Program Equivalence** (**kleene**, **Applicative**, ...)

$$=_{\mathsf{Kl}} = a^{\mathsf{E}\circ}; a^{\mathsf{E}}$$

$$\gtrapprox = \nu x.(a^{\mathsf{E}}; \overline{\Delta \gg x})/a^{\mathsf{E}}$$

**Static Semantics**

Peliminary Results (type elaboration as evaluation)

***Theorem*** $\gtrapprox$ is a congruence: $\widetilde{\gtrapprox} \subseteq \gtrapprox$

*Proof.* Prove $\gtrapprox^{\mathsf{H}} ; a^{\mathsf{E}} \subseteq a^{\mathsf{E}} ; \overline{\Delta \gg \gtrapprox^{\mathsf{H}}}$

$$
\begin{array}{ll}
\Box a^{\S} ; b ; c & \\
\quad \leq \Box(\Diamond a ; \widehat{a^{\S}}) ; b ; c & (\S 3) \\
\quad \leq \Box \Diamond a ; \Box \widehat{a^{\S}} ; b ; c & (\mathrm{K1}) \\
\quad = a ; \Box \widehat{a^{\S}} ; b ; c & (\mathrm{K19}) \text{ since } a = \Box a \\
\quad = a ; \Box \widehat{a^{\S}} ; \Box b ; c & \Box b = b \\
\quad = a ; \Box(\widehat{a^{\S}} ; b) ; c & (\mathrm{K1}) \\
\quad \leq a ; \Box(\widehat{a^{\S}} ; \langle K, \Delta \rangle ; b) ; c & \mathrm{Inv} \\
\quad \leq a ; \Box(\langle a^{\S}, a^{\S} \rangle ; \langle K, \Delta \rangle ; b) ; c & (\mathrm{C19}) \\
\quad \leq a ; \Box \langle a^{\S} ; K, a^{\S} \rangle ; b ; c & (\mathrm{C4}) \\
\quad = a ; \Box \Box \langle a^{\S} ; K, a^{\S} \rangle ; b ; c & (\mathrm{K7}) \\
\quad \leq a ; \Box \langle \Box a^{\S} ; \Box K, \Box a^{\S} \rangle ; \Box b ; c & (\mathrm{C8}) \text{ and } (\mathrm{K1}) \\
\quad = a ; \Box \langle \Box a^{\S} ; K, \Box a^{\S} \rangle ; b ; c & \Box K = K \text{ and } \Box b = b \\
\quad \leq a ; \Box \langle b^{\mathrm{E}} ; \widehat{a^{\S}}, \Box a^{\S} \rangle ; b ; c & \Theta_1 \\
\quad = a ; \Box \langle b^{\mathrm{E}}, \Delta \rangle ; \langle \widehat{a^{\S}}, \Box a^{\S} \rangle ; b ; c & (\mathrm{C4}) \\
\quad \leq a ; \Box \langle b^{\mathrm{E}} \rangle ; \langle \widehat{a^{\S}}, a^{\S} \rangle ; b ; c & (\mathrm{K6}) \text{ and } \langle b^{\mathrm{E}} \rangle = \langle b^{\mathrm{E}}, \Delta \rangle \\
\quad \leq a ; \Box \langle b^{\mathrm{E}} \rangle ; b ; a^{\S}[a^{\S}] ; c & (\mathrm{Harmony}), \text{ since } \widehat{a^{\S}} \leq \widehat{a^{\S}} \\
\quad \leq a ; \Box \langle b^{\mathrm{E}} \rangle ; b ; a^{\S} ; c & (\S 2) \\
\quad \leq a ; \Box \langle b^{\mathrm{E}} \rangle ; \Box b ; \Box a^{\S} ; c & (\mathrm{K1}) \\
\quad \leq a ; \langle b^{\mathrm{E}} \rangle ; b ; \Box a^{\S} ; c & (\mathrm{C17}) \text{ and } \Box b^{\mathrm{E}} = b \text{ and } \Box b = b \\
\quad \leq a ; \langle b^{\mathrm{E}} \rangle ; b ; b^{\mathrm{E}} ; \widehat{a^{\S}} & \ddagger \\
\quad \leq a ; b^{\mathrm{E}} ; \widehat{a^{\S}} & \text{Definition of } b^{\mathrm{E}} \\
\quad \leq b^{\mathrm{E}} ; \widehat{\Diamond a} ; \widehat{a^{\S}} & \dagger \\
\quad \leq b^{\mathrm{E}} ; \widehat{\Diamond a ; a^{\S}} & (\mathrm{E25}) \\
\quad \leq b^{\mathrm{E}} ; \widehat{a^{\S}} & (\mathrm{H17})
\end{array}
$$

# Conclusion

# I Suspect I Am Late...

$$Syn \xrightarrow{\textbf{Syntax} \rightarrow \textbf{Semantics}} \mathit{Rel}(Syn, Syn)$$

$$\textbf{points}(\mathfrak{A}) \xleftarrow{\textbf{Syntax} \leftarrow \textbf{Semantics}} \mathfrak{A}$$

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A rich **research agenda**

2023: *Allegories of Symbolic Manipulation*

Now: *Allegories of Operational Semantics*

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Now: *Allegories of Operational Semantics*

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Future *Allegories of Static Semantics*

Future *Allegories of Computational Effects*

Future *Allegories of Computational Coeffects*

Future *Allegories of Mechanised Semantics*