Università di Genova | DIMA DIPARTIMENTO DI MATEMATICA

Fibrations with comprehension and their completion

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Comprehension categories & Lawvere-Ehrhard comprehension

Our work: completion of both the structures and comparison between them

Why: better understanding of comprehensions, easier construction of models

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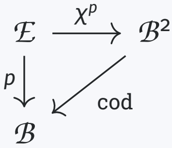
Comprehension categories

Definition

A **comprehension category** is a pair (p, χ^p) with $p\colon \mathcal{E} \to \mathcal{B}$ a fibration and $\chi^p\colon \mathcal{E} \to \mathcal{B}^2$ such that $\mathrm{cod} \circ \chi^p = p$ and that χ^p preserves cartesian arrows



First examples

Example

The pair $(\mathrm{cod}, \mathrm{Id}_{\mathcal{C}^2})$ for a category $\mathcal{C}$ with pullbacks. More in general their restriction to any full subcategory closed under pullbacks.

Example

The pair $(\mathrm{Fam}_{\mathcal{S}et}, \chi)$ where $\mathrm{Fam}_{\mathcal{S}et}\colon \mathrm{Fam}(\mathcal{S}et) \to \mathcal{S}et$ is the family fibration of sets and

$$\chi(I, \{X_i\}_{i\in I}) := \coprod_{i\in I} X_i \to I$$

Example

The syntactic fibration associated with a dependent type theory.

Comprehension categories completion

Let p be a fibration. Consider the pullback in $\mathcal{C}at$

$$\begin{array}{ccc} \mathcal{E}^p & \longrightarrow & \mathcal{E} \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{FFP}(p) & \xrightarrow[\mathit{ffp}(p)]{} & \mathcal{B} \end{array}$$

where $\mathit{ffp}(p)$ is the free fibration with finite fibred products.

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where $\mathit{ffp}(p)$ is the free fibration with finite fibred products.

Proposition

$\hat{p}$ can be endowed with the structure of comprehension category.

Sketch of proof

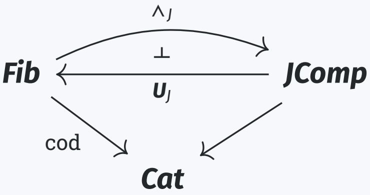
The comprehension functor $\chi^p \colon \mathcal{E}^p \to \mathcal{FFP}(p)^2$ is given by $((X, \vec{A}), A) \mapsto ((X, \vec{A}, A) \to (X, \vec{A}))$, where $A, A_i \in \mathcal{E}_X$ and $X \in \mathcal{B}$. □

Comprehension categories completion

The construction extends to a 2-functor $\wedge_J$.

Theorem

The 2-functor $\wedge_J$ is left bi-adjoint to the forgetful 2-functor $\mathbf{U}_{JC}$.



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Lawvere-Ehrhard comprehension

Lawvere: hyperdoctrines satisfying Comprehension Schema, i.e. a bifibration whose basis and fibers are cartesian closed, there are a right adjoint to substitution and a right adjoint to the existential quantification of fibered terminals. [1]

Ehrhard: D-categories, i.e. fibrations p with a RARI (right adjoint-right inverse) T and a right adjoint to it C. [2]

Jacobs calls them “comprehension categories with unit”. [3]

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[1] *F.W.Lawvere, Equality in hyperdoctrines and comprehension schema as an adjoint functor, Applications of Categorical Algebra, 1970*

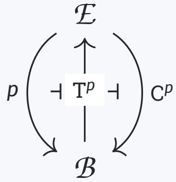
[2] *T.Ehrhard, A categorical semantics of constructions, in Proceedings. Third Annual Symposium on Logic in Computer Science, 1988*

[3] *B.Jacobs, Comprehension categories and the semantics of type dependencies, in Theoretical Computer Science, 1993*

Lawvere-Ehrhard comprehension

Definition

A **fibration with Lawvere-Ehrhard comprehension** is a triple $(p, \mathrm{T}^p, \mathrm{C}^p)$ where $p\colon \mathcal{E} \to \mathcal{B}$ is a fibration, $\mathrm{T}^p\colon \mathcal{B} \to \mathcal{E}$ is right adjoint and right inverse (RARI) to p, and C^p is right adjoint to T^p



First examples

Example

The triple $\mathcal{C}^2 \; \substack{\xrightarrow{\mathrm{dom}} \\ \top \\ \xleftarrow{\mathrm{T}^{\mathrm{cod}}} \\ \top \\ \xrightarrow[\mathrm{cod}]{}} \; \mathcal{C}$ for a category $\mathcal{C}$ with pullbacks, where $\mathrm{T}^{\mathrm{cod}}C := \mathrm{id}_C$

Example

The triple $\mathcal{Pred} \; \substack{\xrightarrow{\mathrm{C}^{\mathrm{Pr}}} \\ \top \\ \xleftarrow{\mathrm{T}^{\mathrm{Pr}}} \\ \top \\ \xrightarrow[\mathrm{Pr}]{}} \; \mathcal{Set}$ where Pr is the first projection,

$\mathrm{T}^{\mathrm{Pr}}X := (X, X)$ and C^{Pr} is the second projection on objects

Example

The full syntactic fibration associated with a dependent type theory with $\mathbf{1}$

Another example: the simple fibration

Let $\mathcal{B}$ be a category with finite products $\times$

Definition

Consider the comonad induced by the functor $T\colon \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ defined by

$$T(I, X) = (I, I \times X)$$

Denote its coKleisli category as $\mathrm{s}(\mathcal{B})$. An arrow $(I, X) \to (J, Y)$ in $\mathrm{s}(\mathcal{B})$ corresponds to arrows $I \to J$ and $I \times X \to Y$ in $\mathcal{B}$.

The first projection on both objects and arrows $\mathrm{s}_{\mathcal{B}}\colon \mathrm{s}(\mathcal{B}) \to \mathcal{B}$ is a fibration which is called the **simple fibration** on $\mathcal{B}$

Lawvere-Ehrhard completion

Let p be a fibration with finite fibred products.

Definition

Consider the fibration $q\colon \mathcal{E}' \to \mathcal{E}$ obtained by pullback of p along itself.

Consider the comonad induced by the functor $T\colon \mathcal{E}' \to \mathcal{E}'$ defined by

$$T(A, B) = (A, A \wedge B)$$

where $A, B \in \mathcal{E}_X$.

Denote its coKleisli category as $\mathcal{E}^p$.

The completion $\hat{p}\colon \mathcal{E}^p \to \mathcal{E}$ is the first projection

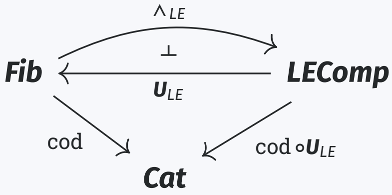
Properties of the completion

Proposition

The completion $\wedge_{LE}$ preserves faithfulness.

Proposition

The completion $\wedge_{LE}$ preserves cartesian arrows of ***Fib****.*



Completion and simple fibration

There is a relation between $\wedge_{LE}$ and the simple fibration.

Remark

The completion applied to the terminal fibration $!_{\mathcal{B}} \colon \mathcal{B} \to \mathbf{1}$ yields the simple fibration $\mathsf{s}_{\mathcal{B}} \colon \mathsf{s}(\mathcal{B}) \to \mathcal{B}$

Corollary

The completion acts on fibers as the simple fibration construction

$$
\begin{CD} \mathcal{E}_X @>>> \mathcal{E} \\ @V{!}VV @VV{p}V \\ \mathbf{1} @>>{X}> \mathcal{B} \end{CD} \qquad \longmapsto \qquad \begin{CD} \mathsf{s}(\mathcal{E}_X) @>>> \mathcal{E}^p \\ @V{\mathsf{s}_{\mathcal{E}_X}}VV @VV{\hat{p}}V \\ \mathcal{E}_X @>>> \mathcal{E} \end{CD}
$$

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Lawvere-Ehrhard comprehension implies Jacobs'

Let $(p, \mathrm{T}, \mathrm{C})$ be a fibration with Lawvere-Ehrhard comprehension. Define $\chi\colon \mathcal{E} \to \mathcal{B}^2$ by $\chi A := p\epsilon_A$, where ϵ is the counit of $\mathrm{T} \dashv \mathrm{C}$.

Theorem

(p, χ) is a comprehension category. Furthermore the assignation extends to a 2-functor ***LE-J***: ***LEComp*** $\to$ ***JComp***.

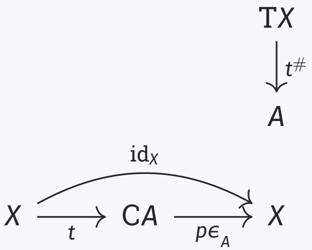
Correspondence between terms and proof terms

Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere-Ehrhard comprehension.

Proposition

The terms of p correspond bijectively to global proof terms via transposition.

$$\mathrm{Hom}_{\mathcal{B}}(X, \mathrm{C}A) \cong \mathrm{Hom}_{\mathcal{E}}(\mathrm{T}X, A)$$



Essential image characterization

Theorem

*Let $p\colon \mathcal{E} \to \mathcal{B}$ be a comprehension category together with a RARI T. Then p is in the essential image of **LE-J** if and only if:*

Given X in $\mathcal{B}$, there is a section $s_X\colon X \to \mathrm{CT}X$ of $\chi\mathrm{T}X$;

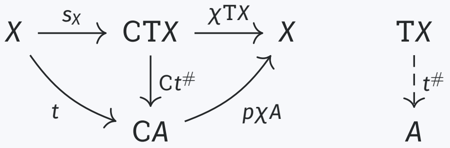
Essential image characterization

Theorem

*Let $p\colon \mathcal{E} \to \mathcal{B}$ be a comprehension category together with a RARI T. Then p is in the essential image of **LE-J** if and only if:*

Given X in $\mathcal{B}$, there is a section $s_X\colon X \to \mathrm{C}\mathrm{T}X$ of $\chi\mathrm{T}X$;

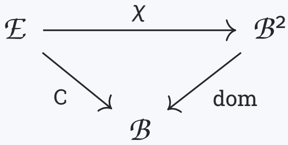
Given A over X and a section $t\colon X \to \mathrm{C}A$ of χA, there exist a unique vertical arrow $t^{\#}\colon \mathrm{T}X \to A$ such that $\mathrm{C}t^{\#} \circ s_X = t$.



An application of the characterization

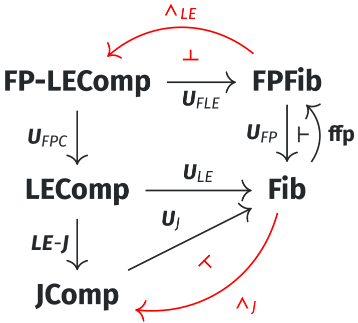
Corollary

Let $p\colon \mathcal{E} \to \mathcal{B}$ be a full comprehension category with fibred terminals such that χ preserves them. Then p together with $\mathrm{dom} \circ \chi$ is a fibration with Lawvere-Ehrhard comprehension.



Conclusions

- Bi-adjunction $\boldsymbol{U}_{FLC} \vdash \wedge_{LE}$
- Bi-adjunction $\boldsymbol{U}_{JC} \vdash$ **JComp**
- Characterization of the essential image of ***LE-J***



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- Bi-adjunction $U_{FLC} \vdash \wedge_{LE}$
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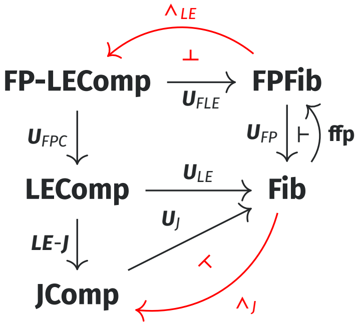
- Characterization of the essential image of ***LE-J***



Further developments:

- investigate missing universal constructions as a left adjoint to ***LE-J*** and U_{LC}
- investigate monadicity of the constructions we provided
- investigate structures preserved by our constructions

Thank you for your attention!

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