GATs, Cats, and CwFs

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What is this talk?

- Background for my forthcoming PhD thesis [Neu25, Chapter 1]
- Interpretation of [KKA19]
  - Restrict their signature language for QIITs to get a signature language for GATs
  - Matches the notion of GAT given by [Car86], except we don't allow for equations between sort. This provides an intrinsic presentation of GATs.
- Slides
  - jacobneu.com/WG6
  - jacobneu.com/WG6-extended

	GAT signature	Category of Algs.	Concrete CwF
Categories			
CwFs			
GAT signatures			

Categories of GAT Algebras

$$
\begin{array}{l}
\textsf{def}\ \mathfrak{Cat} : \mathtt{GAT} := \{\!\!\{ \\
\quad \mathtt{Obj} \quad : \mathbf{U}, \\
\quad \mathtt{Hom} \quad : \mathtt{Obj} \Rightarrow \mathtt{Obj} \Rightarrow \mathbf{U}, \\
\quad \mathtt{id} \quad\ \ : (\mathtt{I} : \mathtt{Obj}) \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{I}, \\
\quad \mathtt{comp} \ : \{\mathtt{I}\ \mathtt{J}\ \mathtt{K} : \mathtt{Obj}\} \Rightarrow \\
\qquad\qquad \mathtt{Hom}\ \mathtt{J}\ \mathtt{K} \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{J} \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{K},
\end{array}
$$

$$
\begin{aligned}
&\texttt{lunit} : \{\texttt{I J} : \texttt{Obj}\} \Rightarrow (\texttt{j} : \texttt{Hom I J}) \Rightarrow \\
&\qquad \texttt{comp (id J) j} \equiv \texttt{j}, \\
&\texttt{runit} : \{\texttt{I J} : \texttt{Obj}\} \Rightarrow (\texttt{j} : \texttt{Hom I J}) \Rightarrow \\
&\qquad \texttt{comp j (id I)} \equiv \texttt{j}, \\
&\texttt{assoc} : \{\texttt{I J K L} : \texttt{Obj}\} \Rightarrow (\texttt{j} : \texttt{Hom I J}) \Rightarrow \\
&\qquad (\texttt{k} : \texttt{Hom J K}) \Rightarrow (\ell : \texttt{Hom K L}) \Rightarrow \\
&\qquad \texttt{comp}\ \ell\ (\texttt{comp k j}) \equiv \texttt{comp}\ (\texttt{comp}\ \ell\ \texttt{k})\ \texttt{j}
\end{aligned}
$$

$\rrbracket$

	GAT signature	Category of Algs.
Categories	$\mathfrak{Cat}$	Cat

$$
\begin{array}{l}
\text{def } \mathfrak{CwF} : \mathtt{GAT} := \{\!\!\{ \\
\quad \text{include } \mathfrak{Cat} \text{ as } (\mathtt{Con}, \mathtt{Sub}, \mathtt{comp}, \mathtt{id}, \_, \_, \_); \\
\qquad \mathtt{empty} \quad : \mathtt{Con}, \\
\qquad \epsilon \qquad\quad : (\Gamma : \mathtt{Con}) \Rightarrow \mathtt{Sub}\ \Gamma\ \mathtt{empty}, \\
\qquad \epsilon\_\eta \qquad : (\Gamma : \mathtt{Con}) \Rightarrow (\mathtt{f} : \mathtt{Sub}\ \Gamma\ \mathtt{empty}) \Rightarrow \\
\qquad\qquad \mathtt{f} \equiv (\epsilon\ \Gamma),
\end{array}
$$

$$
\begin{aligned}
&\mathtt{Ty} && : \mathtt{Con} \Rightarrow \mathbf{U},\\
&\mathtt{substTy} && : \{\Delta\ \Gamma : \mathtt{Con}\} \Rightarrow\\
& && \quad \mathsf{Sub}\ \Delta\ \Gamma \Rightarrow \mathtt{Ty}\ \Gamma \Rightarrow \mathtt{Ty}\ \Delta,\\
&\mathtt{idTy} && : \{\Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma) \Rightarrow\\
& && \quad \mathtt{substTy}\ (\mathtt{id}\ \Gamma)\ \mathtt{A} \equiv \mathtt{A},\\
&\mathtt{compTy} && : \{\Theta\ \Delta\ \Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma)\\
& && \quad (\delta : \mathsf{Sub}\ \Theta\ \Delta) \Rightarrow (\gamma : \mathsf{Sub}\ \Delta\ \Gamma) \Rightarrow\\
& && \quad \mathtt{substTy}\ \gamma\ (\mathtt{substTy}\ \delta\ \mathtt{A})\\
& && \quad \equiv \mathtt{substTy}\ (\mathtt{comp}\ \gamma\ \delta)\ \mathtt{A},
\end{aligned}
$$

$$\begin{aligned}
&\mathtt{Tm} \quad\quad\;\, : (\Gamma : \mathtt{Con}) \Rightarrow \mathtt{Ty}\ \Gamma \Rightarrow \mathbf{U},\\
&\mathtt{substTm} : \{\Delta\ \Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow\\
&\qquad (\gamma : \mathtt{Sub}\ \Delta\ \Gamma) \Rightarrow\\
&\qquad \mathtt{Tm}\ \Gamma\ \mathtt{A} \Rightarrow \mathtt{Tm}\ \Delta\ (\mathtt{substTy}\ \gamma\ \mathtt{A}),
\end{aligned}$$

$$
\begin{aligned}
&\mathtt{idTm} \quad : \{\Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow (\mathtt{t} : \mathtt{Tm}\ \Gamma\ \mathtt{A}) \\
&\qquad \mathtt{substTm}\ (\mathtt{id}\ \Gamma)\ \mathtt{t} \qquad \#\langle \mathtt{idTy}\ \mathtt{A} \rangle \\
&\qquad \equiv \mathtt{t}, \\
&\mathtt{compTm} \ : \{\Theta\ \Delta\ \Gamma : \mathtt{Con}\} \Rightarrow \\
&\qquad \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow (\mathtt{t} : \mathtt{Tm}\ \Gamma\ \mathtt{A}) \Rightarrow \\
&\qquad (\delta : \mathtt{Sub}\ \Theta\ \Delta) \Rightarrow (\gamma : \mathtt{Sub}\ \Delta\ \Gamma) \Rightarrow \\
&\qquad \mathtt{substTm}\ \gamma\ (\mathtt{substTm}\ \delta\ \mathtt{t}) \\
&\qquad\qquad \#\langle \mathtt{compTy}\ \mathtt{A}\ \gamma\ \delta \rangle \\
&\qquad \equiv \mathtt{substTm}\ (\mathtt{comp}\ \gamma\ \delta)\ \mathtt{t},
\end{aligned}
$$

$$
\begin{aligned}
&\texttt{ext} \quad : (\Gamma : \texttt{Con}) \Rightarrow \texttt{Ty}\ \Gamma \Rightarrow \texttt{Con},\\
&\texttt{pair} \quad : \{\Delta\ \Gamma : \texttt{Con}\} \Rightarrow \{\texttt{A} : \texttt{Ty}\ \Gamma\} \Rightarrow\\
&\qquad (\gamma : \texttt{Sub}\ \Delta\ \Gamma) \Rightarrow\\
&\qquad \texttt{Tm}\ \Delta\ (\texttt{substTy}\ \gamma\ \texttt{A}) \Rightarrow\\
&\qquad \texttt{Sub}\ \Delta\ (\texttt{ext}\ \Gamma\ \texttt{A}),
\end{aligned}
$$

$$
\begin{aligned}
&\mathtt{pair\_nat}\colon \{\Theta\ \Delta\ \Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow \\
&\qquad (\gamma : \mathtt{Sub}\ \Delta\ \Gamma) \Rightarrow \\
&\qquad (\mathtt{t} : \mathtt{Tm}\ \Delta\ (\mathtt{substTy}\ \gamma\ \mathtt{A})) \Rightarrow \\
&\qquad (\delta : \mathtt{Sub}\ \Theta\ \Delta) \Rightarrow \\
&\qquad \mathtt{comp}\ (\mathtt{pair}\ \gamma\ \mathtt{t})\ \delta \\
&\qquad \equiv \mathtt{pair}\ (\mathtt{comp}\ \gamma\ \delta) \\
&\qquad\quad (\mathtt{substTm}\ \delta\ \mathtt{t}\ \ \#\langle \mathtt{compTy}\ \mathtt{A}\ \gamma\ \delta \rangle),
\end{aligned}
$$

$$
\begin{aligned}
&\mathtt{p} \quad : \{\Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma) \Rightarrow \\
&\qquad \mathtt{Sub}\ (\mathtt{ext}\ \Gamma\ \mathtt{A})\ \Gamma \\
&\mathtt{v} \quad : \{\Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma) \Rightarrow \\
&\qquad \mathtt{Tm}\ (\mathtt{ext}\ \Gamma\ \mathtt{A})\ (\mathtt{substTy}\ (\mathtt{p}\ \mathtt{A})\ \mathtt{A}),
\end{aligned}
$$

Categories with Families

```
ext_β  : (Δ Γ : Con) ⇒ (A : Ty Γ) ⇒
     (γ : Sub Δ Γ) ⇒
     (t : Tm Δ (substTy γ A)) ⇒
     comp (p A) (pair γ t) ≡ γ,
ext_β  : (Δ Γ : Con) ⇒ (A : Ty Γ) ⇒
     (γ : Sub Δ Γ) ⇒
     (t : Tm Δ (substTy γ A)) ⇒
     substTm (pair γ t) (v A)
        #⟨compTy A (p A) (pair γ t)⟩
        #⟨ext_β  γ t⟩
     ≡ t,
ext_η  : (Γ : Con) ⇒ (A : Ty Γ) ⇒
     pair (p A) (v A)
     ≡ id (ext Γ A)
```

	GAT signature	Category of Algs.
Categories	$\mathfrak{Cat}$	Cat
CwFs	$\mathfrak{CwF}$	CwF

EXTEND

```
def ℭ𝔴𝔉++ : GAT := {[
  include ℭ𝔴𝔉;
    V : {Γ : Con} ⇒ Ty Γ,
    El : {Γ : Con} ⇒ Tm Γ V ⇒ Ty Γ,
    ...
    Eq : {Γ : Con} ⇒ {A : Ty Γ} ⇒
        (t t' : Tm Γ A) ⇒ Ty Γ,
    ...
    Pi : {Γ : Con} ⇒ (X : Tm Γ V) ⇒
        Ty (ext Γ (El X)) ⇒ Ty Γ,
    ...
```

Why this specific GAT?

1

The GAT Signature Language

We assume that the GAT $\mathfrak{CwF}+\mathsf{V}+\mathsf{Eq}+\Pi_{sd}$ has an initial algebra—the language we'll call $\textsc{one}\mathrm{GAT}$—given as a QIIT in the metatheory.

$$
\begin{aligned}
\mathsf{GAT} := \overline{\mathsf{Con}} &: \mathsf{Set} \\
\overline{\mathsf{Sub}} &: \overline{\mathsf{Con}} \to \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Ty}} &: \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Tm}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to \overline{\mathsf{Ty}}\ \mathfrak{G}
\end{aligned}
$$

Use the DSL parsing utilities in $\textsc{Lean}4$ to translate the intuitive syntax for GATs into $\textsc{one}\mathrm{GAT}$.

$$\begin{array}{ll}
\textsf{def}\ \mathfrak{N} : \mathtt{GAT} := \{\!\![& \diamond \\
\quad \mathtt{Nat} \quad : \mathbf{U}, & \triangleright\ \mathsf{V} \\
\quad \mathtt{zero} \ : \mathtt{Nat}, & \triangleright\ \mathsf{El}\ 0 \\
\quad \mathtt{succ} \ : \mathtt{Nat} \Rightarrow \mathtt{Nat} & \triangleright\ \Pi\ 1\ (\mathsf{El}\ 2) \\
]\!\!\} &
\end{array}$$

Categories

$$
\begin{aligned}
&\mathsf{def}\ \mathfrak{Cat} : \mathtt{GAT} := \{\!\!\{ \\
&\quad \mathtt{Obj} \ : \mathbf{U}, \\
&\quad \mathtt{Hom} \ : \mathtt{Obj} \Rightarrow \mathtt{Obj} \Rightarrow \mathbf{U}, \\
&\quad \mathtt{id} \ : (\mathtt{I} : \mathtt{Obj}) \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{I}, \\
&\quad \mathtt{comp} \ : \{\mathtt{I}\ \mathtt{J}\ \mathtt{K} : \mathtt{Obj}\} \Rightarrow \\
&\qquad\quad \mathtt{Hom}\ \mathtt{J}\ \mathtt{K} \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{J} \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{K}, \\
&\quad \mathtt{lunit} : \{\mathtt{I}\ \mathtt{J} : \mathtt{Obj}\} \Rightarrow (\mathtt{j} : \mathtt{Hom}\ \mathtt{I}\ \mathtt{J}) \Rightarrow \\
&\qquad\quad \mathtt{comp}\ (\mathtt{id}\ \mathtt{J})\ \mathtt{j} \equiv \mathtt{j}, \\
&\quad \mathtt{runit} : \{\mathtt{I}\ \mathtt{J} : \mathtt{Obj}\} \Rightarrow (\mathtt{j} : \mathtt{Hom}\ \mathtt{I}\ \mathtt{J}) \Rightarrow \\
&\qquad\quad \mathtt{comp}\ \mathtt{j}\ (\mathtt{id}\ \mathtt{I}) \equiv \mathtt{j}, \\
&\quad \mathtt{assoc} : \{\mathtt{I}\ \mathtt{J}\ \mathtt{K}\ \mathtt{L} : \mathtt{Obj}\} \Rightarrow \\
&\qquad\quad (\mathtt{j} : \mathtt{Hom}\ \mathtt{I}\ \mathtt{J}) \Rightarrow (\mathtt{k} : \mathtt{Hom}\ \mathtt{J}\ \mathtt{K}) \Rightarrow \\
&\qquad\quad (\ell : \mathtt{Hom}\ \mathtt{K}\ \mathtt{L}) \Rightarrow \\
&\qquad\quad \mathtt{comp}\ \ell\ (\mathtt{comp}\ \mathtt{k}\ \mathtt{j}) \\
&\qquad\quad \equiv \mathtt{comp}\ (\mathtt{comp}\ \ell\ \mathtt{k})\ \mathtt{j} \\
&\}\!\!\}
\end{aligned}
$$

◇

- ▷ V
- ▷ Π 0 (Π 1 V)
- ▷ Π 1 (El (1 @ 0 @ 0))
- ▷ Π 2 (Π 3 (Π 4 (Π (4 @ 1 @ 0) (Π (5 @ 3 @ 2) (El (6 @ 4 @ 2))))))
- ▷ Π 3 (Π 4 (Π (4 @ 1 @ 0) (Eq (3 @ 2 @ 1 @ 1 @ (4 @ 1) @ 0) 0)))
- ▷ Π 4 (Π 5 (Π (5 @ 1 @ 0) (Eq (4 @ 2 @ 2 @ 1 @ 0 @ (5 @ 2)) 0)))
- ▷ Π 5 (Π 6 (Π 7 (Π 8 (Π (8 @ 3 @ 2) (Π (9 @ 3 @ 2) (Π (10 @ 3 @ 2) (Eq (9 @ 6 @ 5 @ 3 @ 0 @ (9 @ 6 @ 5 @ 4 @ 1 @ 2)) (9 @ 6 @ 4 @ 3 @ (9 @ 5 @ 4 @ 3 @ 0 @ 1) @ 2))))))))

CwFs

$\mathfrak{Cat}$
▷ El 6
▷ Π 7 (El (7 @ 0 @ 1))
▷ Π 8 (Π (8 @ 0 @ 2) (Eq 0 (2 @ 1)))
▷ Π 9 U
▷ Π 10 (Π 11 (Π (11 @ 1 @ 0) (Π (3 @ 1) (El (4 @ 3)))))
▷ Π 11 (Π (2 @ 0) (Eq (2 @ 1 @ 1 @ (11 @ 1) @ 0) 0))
▷ Π 12 (Π 13 (Π 14 (Π (5 @ 0) (Π (15 @ 3 @ 2) (Π (16 @ 3 @ 2) (Eq (7 @ 4 @ 3 @ 0 @ (7 @ 5 @ 4 @ 1 @ 2)) (7 @ 5 @ 3 @ (15 @ 5 @ 4 @ 3 @ 0 @ 1) @ 2)))))))
▷ Π 13 (Π (4 @ 0) U)
▷ Π 14 (Π 15 (Π (6 @ 0) (Π (16 @ 2 @ 1) (Π (4 @ 2 @ 1) (El (5 @ 4 @ (8 @ 4 @ 3 @ 1 @ 2)))))))
▷ Π 15 (Π (6 @ 0) (Π (3 @ 1 @ 0) (Eq (transp (6 @ 2 @ 1) (3 @ 2 @ 2 @ 1 @ (16 @ 2) @ 0)) 0)))
▷ Π 16 (Π 17 (Π 18 (Π (9 @ 0) (Π (6 @ 1 @ 0) (Π (20 @ 4 @ 3) (Π (21 @ 4 @ 3) (Eq (transp (10 @ 6 @ 5 @ 4 @ 3 @ 0 @ 1) (8 @ 5 @ 4 @ 3 @ 0 @ (8 @ 6 @ 5 @ (12 @ 5 @ 4 @ 0 @ 3) @ 1 @ 2))) (8 @ 6 @ 4 @ 3 @ (20 @ 6 @ 5 @ 4 @ 0 @ 1) @ 2))))))))

CwFs

- Π 17 (Π (8 @ 0) (El 19))
- Π 18 (Π 19 (Π (10 @ 0) (Π (20 @ 2 @ 1) (Π (8 @ 3 @ (11 @ 3 @ 2 @ 0 @ 1)) (El (22 @ 4 @ (5 @ 3 @ 2)))))))
- Π 19 (Π 20 (Π 21 (Π (12 @ 0) (Π (22 @ 2 @ 1) (Π (10 @ 3 @ (13 @ 3 @ 2 @ 0 @ 1)) (Π (24 @ 5 @ 4) (Eq (23 @ 6 @ 5 @ (8 @ 4 @ 3) @ (7 @ 5 @ 4 @ 3 @ 2 @ 1) @ 0) (7 @ 6 @ 4 @ 3 @ (23 @ 6 @ 5 @ 4 @ 2 @ 0) @ (transp (13 @ 6 @ 5 @ 4 @ 3 @ 2 @ 0) (11 @ 6 @ 5 @ (15 @ 5 @ 4 @ 2 @ 3) @ 0 @ 1))))))))))
- Π 20 (Π (11 @ 0) (El (21 @ (4 @ 1 @ 0) @ 1)))
- Π 21 (Π (12 @ 0) (El (9 @ (5 @ 1 @ 0) @ (12 @ (5 @ 1 @ 0) @ 1 @ (2 @ 1 @ 0) @ 0))))
- Π 22 (Π 23 (Π (14 @ 0) (Π (24 @ 2 @ 1) (Π (12 @ 3 @ (15 @ 3 @ 2 @ 0 @ 1)) (Eq (24 @ 4 @ (9 @ 3 @ 2) @ 3 @ (6 @ 3 @ 2) @ (8 @ 4 @ 3 @ 2 @ 1 @ 0)) 1)))))
- Π 23 (Π 24 (Π (15 @ 0) (Π (25 @ 2 @ 1) (Π (13 @ 3 @ (16 @ 3 @ 2 @ 0 @ 1)) (Eq (transp (5 @ 4 @ 3 @ 2 @ 1 @ 0) (transp (15 @ 4 @ (10 @ 3 @ 2) @ 3 @ 2 @ (7 @ 3 @ 2) @ (9 @ 4 @ 3 @ 2 @ 1 @ 0)) (13 @ 4 @ (10 @ 3 @ 2) @ (17 @ 4 @ 3 @ 1 @ 2) @ (9 @ 4 @ 3 @ 2 @ 1 @ 0) @ (6 @ 3 @ 2)))) 0)))))
- Π 24 (Π (15 @ 0) (Eq (7 @ (8 @ 1 @ 0) @ 1 @ 0 @ (5 @ 1 @ 0) @ (4 @ 1 @ 0)) (24 @ (8 @ 1 @ 0))))

To the logical extreme: oneGAT describes itself

Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.
Given an appropriate *motive* $M_{\overline{\mathsf{Con}}}, M_{\overline{\mathsf{Sub}}}, M_{\overline{\mathsf{Ty}}}, M_{\overline{\mathsf{Tm}}}$ and appropriate *method* m, we get:

$$
\begin{aligned}
&(\mathsf{elim}\ m)_{\overline{\mathsf{Con}}} : (\mathfrak{G} : \mathtt{GAT}) \to M_{\overline{\mathsf{Con}}}(\mathfrak{G}) \\
&(\mathsf{elim}\ m)_{\overline{\mathsf{Sub}}} : (\mathfrak{G}\ \mathfrak{H} : \overline{\mathsf{Con}}) \to (\mathfrak{s} : \overline{\mathsf{Sub}}\ \mathfrak{G}\ \mathfrak{H}) \to \\
&\qquad M_{\overline{\mathsf{Sub}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{H}, \mathfrak{s}) \\
&(\mathsf{elim}\ m)_{\overline{\mathsf{Ty}}} : (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to M_{\overline{\mathsf{Ty}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, \mathcal{X}) \\
&(\mathsf{elim}\ m)_{\overline{\mathsf{Tm}}} : (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to (X : \overline{\mathsf{Tm}}(\mathfrak{G}, \mathcal{X})) \\
&\qquad \to M_{\overline{\mathsf{Tm}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathsf{elim}\ m)_{\overline{\mathsf{Ty}}}\ \mathcal{X}, X)
\end{aligned}
$$

$$\_\text{-}\,\mathsf{Alg} : (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set}$$

$$\mathsf{Hom} : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

Syntax	**Algebras**	
$\Gamma : \mathsf{Con}$	Γ^{A}	$: \mathsf{Set}$
$A : \mathsf{Ty}\,\Gamma$	A^{A}	$: \Gamma^{\mathsf{A}} \to \mathsf{Set}$
$\sigma : \mathsf{Sub}\,\Gamma\,\Delta$	σ^{A}	$: \Gamma^{\mathsf{A}} \to \Delta^{\mathsf{A}}$
$t : \mathsf{Tm}\,\Gamma\,A$	t^{A}	$: (\gamma : \Gamma^{\mathsf{A}}) \to A^{\mathsf{A}}\,\gamma$
$\cdot : \mathsf{Con}$	$\cdot^{\mathsf{A}}$	$:\equiv \top$
$\Gamma \rhd A : \mathsf{Con}$	$(\Gamma \rhd A)^{\mathsf{A}}$	$:\equiv (\gamma : \Gamma^{\mathsf{A}}) \times A^{\mathsf{A}}\,\gamma$
$(A : \mathsf{Ty}\,\Delta)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Ty}\,\Gamma$	$(A[\sigma])^{\mathsf{A}}\,\gamma$	$:\equiv A^{\mathsf{A}}\,(\sigma^{\mathsf{A}}\,\gamma)$
$\mathsf{id} : \mathsf{Sub}\,\Gamma\,\Gamma$	$\mathsf{id}^{\mathsf{A}}\,\gamma$	$:\equiv \gamma$
$(\sigma : \mathsf{Sub}\,\Theta\,\Delta) \circ (\delta : \mathsf{Sub}\,\Gamma\,\Theta) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\sigma \circ \delta)^{\mathsf{A}}\,\gamma$	$:\equiv \sigma^{\mathsf{A}}\,(\delta^{\mathsf{A}}\,\gamma)$
$\epsilon : \mathsf{Sub}\,\Gamma\,\cdot$	$\epsilon^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{tt}$
$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta), (t : \mathsf{Tm}\,\Gamma\,(A[\sigma])) : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)$	$(\sigma, t)^{\mathsf{A}}\,\gamma$	$:\equiv (\sigma^{\mathsf{A}}\,\gamma, t^{\mathsf{A}}\,\gamma)$
$\pi_1\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\pi_1\,\sigma)^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{proj}_1\,(\sigma^{\mathsf{A}}\,\gamma)$
$\pi_2\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Tm}\,\Gamma\,(A[\pi_1\,\sigma])$	$(\pi_2\,\sigma)^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{proj}_2\,(\sigma^{\mathsf{A}}\,\gamma)$
$(t : \mathsf{Tm}\,\Delta\,A)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Tm}\,\Gamma\,(A[\sigma])$	$(t[\sigma])^{\mathsf{A}}\,\gamma$	$:\equiv t^{\mathsf{A}}\,(\sigma^{\mathsf{A}}\,\gamma)$
$[\mathsf{id}] : A[\mathsf{id}] = A$	$[\mathsf{id}]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\mathsf{ass}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{idl} : \mathsf{id} \circ \sigma = \sigma$	$\mathsf{idl}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{idr} : \sigma \circ \mathsf{id} = \sigma$	$\mathsf{idr}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\cdot\eta : \{\sigma : \mathsf{Sub}\,\Gamma\,\cdot\} \to \sigma = \epsilon$	$\cdot\eta^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\rhd\beta_1 : \pi_1\,(\sigma, t) = \sigma$	$\rhd\beta_1{}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\rhd\beta_2 : \pi_2\,(\sigma, t) = t$	$\rhd\beta_2{}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$

$\rhd\eta : (\pi_1\,\sigma, \pi_2\,\sigma) = \sigma$	$\rhd\eta^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$,\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$,\circ^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{U} : \mathsf{Ty}\,\Gamma$	$\mathsf{U}^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{Set}$
$\mathsf{El}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) : \mathsf{Ty}\,\Gamma$	$(\mathsf{El}\,a)^{\mathsf{A}}\,\gamma$	$:\equiv a^{\mathsf{A}}\,\gamma$
$\mathsf{U}[] : \mathsf{U}[\sigma] = \mathsf{U}$	$\mathsf{U}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{El}[] : (\mathsf{El}\,a)[\sigma] = \mathsf{El}\,(a[\sigma])$	$\mathsf{El}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\Pi\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})(B : \mathsf{Ty}\,(\Gamma \rhd \mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\Pi\,a\,B)^{\mathsf{A}}\,\gamma$	$:\equiv (\alpha : a^{\mathsf{A}}\,\gamma) \to B^{\mathsf{A}}\,(\gamma, \alpha)$
$\mathsf{app}\,(t : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B)) : \mathsf{Tm}\,(\Gamma \rhd \mathsf{El}\,a)\,B$	$(\mathsf{app}\,t)^{\mathsf{A}}\,(\gamma, \alpha)$	$:\equiv t^{\mathsf{A}}\,\gamma\,\alpha$
$\Pi[] : (\Pi\,a\,B)[\sigma] = \Pi\,(a[\sigma])\,(B[\sigma^{\uparrow}])$	$\Pi[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{app}[] : (\mathsf{app}\,t)[\sigma \uparrow] = \mathsf{app}\,(t[\sigma])$	$\mathsf{app}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{Id}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})\,(t\,u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\mathsf{Id}\,a\,t\,u)^{\mathsf{A}}\,\gamma$	$:\equiv (t^{\mathsf{A}}\,\gamma = u^{\mathsf{A}}\,\gamma)$
$\mathsf{reflect}\,(e : \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,a\,t\,u)) : t = u$	$(\mathsf{reflect}\,e)^{\mathsf{A}}$	$:\equiv \mathsf{funext}\,e^{\mathsf{A}}$
$\mathsf{Id}[] : (\mathsf{Id}\,a\,t\,u)[\sigma] = \mathsf{Id}\,(a[\sigma])\,(t[\sigma])\,(u[\sigma])$	$\mathsf{Id}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\hat{\Pi}\,(T : \mathsf{Set})\,(B : T \to \mathsf{Ty}\,\Gamma) : \mathsf{Ty}\,\Gamma$	$(\hat{\Pi}\,T\,B)^{\mathsf{A}}\,\gamma$	$:\equiv (\alpha : T) \to (B\,\alpha)^{\mathsf{A}}\,\gamma$
$(t : \mathsf{Tm}\,\Gamma\,(\hat{\Pi}\,T\,B))\,\hat{@}(\alpha : T) : \mathsf{Tm}\,\Gamma\,(B\,\alpha)$	$(t\,\hat{@}\,\alpha)^{\mathsf{A}}\,\gamma$	$:\equiv t^{\mathsf{A}}\,\gamma\,\alpha$
$\hat{\Pi}[] : (\hat{\Pi}\,T\,B)[\sigma] = \hat{\Pi}\,T\,(\lambda\alpha.(B\,\alpha)[\sigma])$	$\hat{\Pi}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\hat{@}[] : (t\,\hat{@}\,\alpha)[\sigma] = (t[\sigma])\,\hat{@}\,\alpha$	$\hat{@}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$

$$(\_)^{A} : (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set}$$

From [KKA19, Appendix A]

$\Gamma : \mathsf{Con}$	Γ^{M}	$: \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}$
$A : \mathsf{Ty}\,\Gamma$	A^{M}	$: \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1 \to A^{\mathsf{A}}\,\gamma^0 \to A^{\mathsf{A}}\,\gamma^1 \mathsf{Set}$
$\sigma : \mathsf{Sub}\,\Gamma\,\Delta$	σ^{M}	$: \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1 \to \Delta^{\mathsf{M}}\,(\sigma^{\mathsf{A}}\,\gamma^0)\,(\sigma^{\mathsf{A}}\,\gamma^1)$
$t : \mathsf{Tm}\,\Gamma\,A$	t^{M}	$: (\gamma^M : \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1) \to A^{\mathsf{M}}\,\gamma^M\,(t^{\mathsf{A}}\,\gamma^0)\,(t^{\mathsf{A}}\,\gamma^1)$
$\cdot : \mathsf{Con}$	$\cdot^{\mathsf{M}}\ \mathsf{tt}\,\mathsf{tt}$	$:\equiv \top$
$\Gamma \rhd A : \mathsf{Con}$	$(\Gamma \rhd A)^{\mathsf{M}}\,(\gamma^0, \alpha^0)\,(\gamma^1, \alpha^1)$	$:\equiv (\gamma^M : \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1) \times A^{\mathsf{M}}\,\gamma^M\,\alpha^0\,\alpha^1$
$(A : \mathsf{Ty}\,\Delta)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Ty}\,\Gamma$	$(A[\sigma])^{\mathsf{M}}\,\gamma^M\,\alpha^0\,\alpha^1$	$:\equiv A^{\mathsf{M}}\,(\sigma^{\mathsf{M}}\,\gamma^M)\,\alpha^0\,\alpha^1$
$\mathsf{id} : \mathsf{Sub}\,\Gamma\,\Gamma$	$\mathsf{id}^{\mathsf{M}}\,\gamma^M$	$:\equiv \gamma^m$
$(\sigma : \mathsf{Sub}\,\Theta\,\Delta) \circ (\delta : \mathsf{Sub}\,\Gamma\,\Theta) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\sigma \circ \delta)^{\mathsf{M}}\,\gamma^M$	$:\equiv \sigma^{\mathsf{M}}\,(\delta^{\mathsf{M}}\,\gamma^M)$
$\epsilon : \mathsf{Sub}\,\Gamma\,\cdot$	$\epsilon^{\mathsf{M}}\,\gamma^M$	$:\equiv \mathsf{tt}$
$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta), (t : \mathsf{Tm}\,\Gamma\,(A[\sigma])) : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)$	$(\sigma, t)^{\mathsf{M}}\,\gamma^M$	$:\equiv (\sigma^{\mathsf{M}}\,\gamma^M, t^{\mathsf{M}}\,\gamma^M)$
$\pi_1\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\pi_1\,\sigma)^{\mathsf{M}}\,\gamma^M$	$:\equiv \mathsf{proj}_1\,(\sigma^{\mathsf{M}}\,\gamma^M)$
$\pi_2\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Tm}\,\Gamma\,(A[\pi_1\,\sigma])$	$(\pi_2\,\sigma)^{\mathsf{M}}\,\gamma^M$	$:\equiv \mathsf{proj}_2\,(\sigma^{\mathsf{M}}\,\gamma^M)$
$(t : \mathsf{Tm}\,\Delta\,A)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Tm}\,\Gamma\,(A[\sigma])$	$(t[\sigma])^{\mathsf{M}}\,\gamma^M$	$:\equiv t^{\mathsf{M}}\,(\sigma^{\mathsf{M}}\,\gamma^M)$
$[\mathsf{id}] : A[\mathsf{id}] = A$	$[\mathsf{id}]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\mathsf{ass}^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{idl} : \mathsf{id} \circ \sigma = \sigma$	$\mathsf{idl}^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{idr} : \sigma \circ \mathsf{id} = \sigma$	$\mathsf{idr}^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\cdot\eta : \{\sigma : \mathsf{Sub}\,\Gamma\,\cdot\} \to \sigma = \epsilon$		
$\rhd\beta_1 : \pi_1\,(\sigma, t) = \sigma$		
$\rhd\beta_2 : \pi_2\,(\sigma, t) = t$		
$\rhd\eta : (\pi_1\,\sigma, \pi_2\,\sigma) = \sigma$	$\rhd\eta^{m}$	$:\equiv \mathsf{refl}$
$,\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$,\circ^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{U} : \mathsf{Ty}\,\Gamma$	$\mathsf{U}^{\mathsf{M}}\,\gamma^M\,T^0\,T^1$	$:\equiv T^0 \to T^1$

$\mathsf{El}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) : \mathsf{Ty}\,\Gamma$	$(\mathsf{El}\,a)^{\mathsf{M}}\,\gamma^M\,\alpha^0\,\alpha^1$	$:\equiv a^{\mathsf{M}}\,\gamma^M\,\alpha^0 = \alpha^1$
$\mathsf{U}[] : \mathsf{U}[\sigma] = \mathsf{U}$	$\mathsf{U}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{El}[] : (\mathsf{El}\,a)[\sigma] = \mathsf{El}\,(a[\sigma])$	$\mathsf{El}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\Pi\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})(B : \mathsf{Ty}\,(\Gamma \rhd \mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\Pi\,a\,B)^{\mathsf{M}}\,\gamma^M\,f^0\,f^1$	$:\equiv (\alpha^0 : a^{\mathsf{A}}\,\gamma^0) \to B^{\mathsf{M}}\,(\gamma^M, \mathsf{refl})\,(f^0\,\alpha^0)\,(f^1\,(a^{\mathsf{M}}\,\gamma^M\,\alpha^0))$
$\mathsf{app}\,(t : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B)) : \mathsf{Tm}\,(\Gamma \rhd \mathsf{El}\,a)\,B$	$(\mathsf{app}\,t)^{\mathsf{M}}\,(\gamma^M, \alpha^M)$	$:\equiv \mathsf{J}\,(t^{\mathsf{M}}\,\gamma^M\,\alpha^0)\,\alpha^M$
$\Pi[] : (\Pi\,a\,B)[\sigma] = \Pi\,(a[\sigma])\,(B[\sigma^{\uparrow}])$	$\Pi[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{app}[] : (\mathsf{app}\,t)[\sigma\uparrow] = \mathsf{app}\,(t[\sigma])$	$\mathsf{app}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{Id}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})\,(t\,u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\mathsf{Id}\,a\,t\,u)^{\mathsf{M}}\,\gamma^M\,e^0\,e^1$	$:\equiv \top$
$\mathsf{reflect}\,(e : \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,a\,t\,u)) : t = u$	$(\mathsf{reflect}\,e)^{\mathsf{M}}$	$:\equiv \mathsf{UIP}$
$\mathsf{Id}[] : (\mathsf{Id}\,a\,t\,u)[\sigma] = \mathsf{Id}\,(a[\sigma])\,(t[\sigma])\,(u[\sigma])$	$\mathsf{Id}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\hat{\Pi}\,(T : \mathsf{Set})\,(B : T \to \mathsf{Ty}\,\Gamma) : \mathsf{Ty}\,\Gamma$	$(\hat{\Pi}\,T\,B)^{\mathsf{M}}\,\gamma^M\,f^0\,f^1$	$:\equiv (\alpha : T) \to (B\,\alpha)^{\mathsf{M}}\,\gamma^M\,(f^0\,\alpha)\,(f^1\,\alpha)$
$(t : \mathsf{Tm}\,\Gamma\,(\hat{\Pi}\,T\,B))\,\hat{@}(\alpha : T) : \mathsf{Tm}\,\Gamma\,(B\,\alpha)$	$(t\,\hat{@}\,\alpha)^{\mathsf{M}}\,\gamma^M$	$:\equiv t^{\mathsf{M}}\,\gamma^M\,\alpha$
$\hat{\Pi}[] : (\hat{\Pi}\,T\,B)[\sigma] = \hat{\Pi}\,T\,(\lambda\alpha.(B\,\alpha)[\sigma])$	$\hat{\Pi}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\hat{@}[] : (t\,\hat{@}\,\alpha)[\sigma] = (t[\sigma])\,\hat{@}\,\alpha$	$\hat{@}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$

$$(\_)^M : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$		
CwFs	$\mathfrak{CwF}$		
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	ONEGAT

Idea:

Elements of initial algebra are *only* those derivable from the *mere concept* of the structure

$$\text{def } \mathfrak{N} : \mathtt{GAT} := \{\!\!\{\ \mathtt{Nat} : \mathsf{U}, \mathtt{zero} : \mathtt{Nat}, \mathtt{succ} : \mathtt{Nat} \Rightarrow \mathtt{Nat}\ \}\!\!\}$$

$$\mathbb{N} := \overline{\mathsf{Tm}}(\mathfrak{N}, \mathtt{Nat})$$

$$\mathtt{Nat} : \mathtt{U}, \mathtt{zero} : \mathtt{Nat}, \mathtt{succ} : \mathtt{Nat} \Rightarrow \mathtt{Nat} \vdash \mathtt{t} : \mathtt{Nat}$$

Forms a $\mathfrak{N}$-algebra with $\mathtt{zero}$ and $\lambda\mathtt{t} \to \mathtt{succ}\ \mathtt{t}$.

Claim Can do this for arbitrary GAT $\mathfrak{G}$, by quotient induction-induction on $\textsc{oneGAT}$.

Upshot: Any GAT extension of $\mathfrak{CwF}$ has an initial model

2

Concrete CwFs

Ahrens and Lumsdaine [AL19] introduce the notion of **displayed categories**

$$
\begin{aligned}
\mathsf{Obj}^{\mathsf{D}} &: \mathsf{Obj} \to \mathsf{Set} \\
\mathsf{Hom}^{\mathsf{D}} &: (I\ J : \mathsf{Obj}) \to \mathsf{Obj}^{\mathsf{D}}\ I \to \mathsf{Obj}^{\mathsf{D}}\ J \to \mathsf{Hom}\ I\ J \to \mathsf{Set} \\
\mathsf{id}^{\mathsf{D}} &: (I : \mathsf{Obj}) \to (I^{\mathsf{D}} : \mathsf{Obj}^{\mathsf{D}}\ I) \to \mathsf{Hom}^{\mathsf{D}}\ I\ I\ I^{\mathsf{D}}\ I^{\mathsf{D}}\ \mathsf{id}_I \\
&\ldots
\end{aligned}
$$

Fact: We can define "displayed $\mathfrak{G}$-algebra" for every GAT $\mathfrak{G}$

$$\_\text{-}\mathsf{Alg} : (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set}$$

$$\mathsf{Hom} : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

$$\_\text{-}\mathsf{DAlg} : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

From [KKA19, Appendix A]

Syntax	Displayed algebras	
$\Gamma : \mathsf{Con}$	Γ^{D}	$: \Gamma^{\mathsf{A}} \to \mathsf{Set}$
$A : \mathsf{Ty}\,\Gamma$	A^{D}	$: \Gamma^{\mathsf{D}}\,\gamma \to A^{\mathsf{A}}\,\gamma \to \mathsf{Set}$
$\sigma : \mathsf{Sub}\,\Gamma\,\Delta$	σ^{D}	$: \Gamma^{\mathsf{D}}\,\gamma \to \Delta^{\mathsf{D}}\,(\sigma^{\mathsf{A}}\,\gamma)$
$t : \mathsf{Tm}\,\Gamma\,A$	t^{D}	$: (\gamma^D : \Gamma^{\mathsf{D}}\,\gamma) \to A^{\mathsf{D}}\,\gamma^D\,(t^{\mathsf{A}}\,\gamma)$
$\cdot : \mathsf{Con}$	$\cdot^{\mathsf{D}}\ \mathsf{tt}$	$:\equiv \top$
$\Gamma \rhd A : \mathsf{Con}$	$(\Gamma \rhd A)^{\mathsf{D}}\,(\gamma, \alpha)$	$:\equiv (\gamma^D : \Gamma^{\mathsf{D}}\,\gamma) \times A^{\mathsf{D}}\,\gamma^D\,\alpha$
$(A : \mathsf{Ty}\,\Delta)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Ty}\,\Gamma$	$(A[\sigma])^{\mathsf{D}}\,\gamma^D\,\alpha$	$:\equiv A^{\mathsf{D}}\,(\sigma^{\mathsf{D}}\,\gamma^D)\,\alpha$
$\mathsf{id} : \mathsf{Sub}\,\Gamma\,\Gamma$	$\mathsf{id}^{\mathsf{D}}\,\gamma^D$	$:\equiv \gamma^D$
$(\sigma : \mathsf{Sub}\,\Theta\,\Delta) \circ (\delta : \mathsf{Sub}\,\Gamma\,\Theta) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\sigma \circ \delta)^{\mathsf{D}}\,\gamma^D$	$:\equiv \sigma^{\mathsf{D}}\,(\delta^{\mathsf{D}}\,\gamma^D)$
$\epsilon : \mathsf{Sub}\,\Gamma\,\cdot$	$\epsilon^{\mathsf{D}}\,\gamma^D$	$:\equiv \mathsf{tt}$
$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta), (t : \mathsf{Tm}\,\Gamma\,(A[\sigma])) : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)$	$(\sigma, t)^{\mathsf{D}}\,\gamma^D$	$:\equiv (\sigma^{\mathsf{D}}\,\gamma^D, t^{\mathsf{D}}\,\gamma^D)$
$\pi_1\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\pi_1\,\sigma)^{\mathsf{D}}\,\gamma^D$	$:\equiv \mathsf{proj}_1\,(\sigma^{\mathsf{D}}\,\gamma^D)$
$\pi_2\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Tm}\,\Gamma\,(A[\pi_1\,\sigma])$	$(\pi_2\,\sigma)^{\mathsf{D}}\,\gamma^D$	$:\equiv \mathsf{proj}_2\,(\sigma^{\mathsf{D}}\,\gamma^D)$
$(t : \mathsf{Tm}\,\Delta\,A)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Tm}\,\Gamma\,(A[\sigma])$	$(t[\sigma])^{\mathsf{D}}\,\gamma^D$	$:\equiv t^{\mathsf{D}}\,(\sigma^{\mathsf{D}}\,\gamma^D)$
$[\mathsf{id}] : A[\mathsf{id}] = A$	$[\mathsf{id}]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\mathsf{ass}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{idl} : \mathsf{id} \circ \sigma = \sigma$	$\mathsf{idl}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{idr} : \sigma \circ \mathsf{id} = \sigma$	$\mathsf{idr}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\cdot\eta : \{\sigma : \mathsf{Sub}\,\Gamma\,\cdot\} \to \sigma = \epsilon$	$\cdot\eta^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\rhd\beta_1 : \pi_1\,(\sigma, t) = \sigma$	$\rhd\beta_1{}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\rhd\beta_2 : \pi_2\,(\sigma, t) = t$	$\rhd\beta_2{}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\rhd\eta : (\pi_1\,\sigma, \pi_2\,\sigma) = \sigma$	$\rhd\eta^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$,\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$,\circ^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{U} : \mathsf{Ty}\,\Gamma$	$\mathsf{U}^{\mathsf{D}}\,\gamma^D\,T$	$:\equiv T \to \mathsf{Set}$

$\mathsf{El}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) : \mathsf{Ty}\,\Gamma$	$(\mathsf{El}\,a)^{\mathsf{D}}\,\gamma^D\,\alpha$	$:\equiv a^{\mathsf{D}}\,\gamma^D\,\alpha$
$\mathsf{U}[] : \mathsf{U}[\sigma] = \mathsf{U}$	$\mathsf{U}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{El}[] : (\mathsf{El}\,a)[\sigma] = \mathsf{El}\,(a[\sigma])$	$\mathsf{El}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\Pi\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})(B : \mathsf{Ty}\,(\Gamma \rhd \mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\Pi\,a\,B)^{\mathsf{D}}\,\gamma^D\,f$	$:\equiv (\alpha^D : a^{\mathsf{D}}\,\gamma^D\,\alpha) \to$ $B^{\mathsf{D}}\,(\gamma^D, \alpha^D)\,(f\,\alpha)$
$\mathsf{app}\,(t : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B)) : \mathsf{Tm}\,(\Gamma \rhd \mathsf{El}\,a)\,B$	$(\mathsf{app}\,t)^{\mathsf{D}}\,(\gamma^D, \alpha^D)$	$:\equiv t^{\mathsf{D}}\,\gamma^D\,\alpha^D$
$\Pi[] : (\Pi\,a\,B)[\sigma] = \Pi\,(a[\sigma])\,(B[\sigma^{\uparrow}])$	$\Pi[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{app}[] : (\mathsf{app}\,t)[\sigma\uparrow] = \mathsf{app}\,(t[\sigma])$	$\mathsf{app}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{Id}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})\,(t\,u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\mathsf{Id}\,a\,t\,u)^{\mathsf{D}}\,\gamma^D\,e$	$:\equiv \mathsf{tr}_{(a^{\mathsf{D}}\,\gamma^D)}\,e\,(t^{\mathsf{D}}\,\gamma^D) = u^{\mathsf{D}}\,\gamma^D$
$\mathsf{reflect}\,(e : \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,a\,t\,u)) : t = u$	$(\mathsf{reflect}\,e)^{\mathsf{D}}$	$: t^{\mathsf{D}}\,\gamma^D \overset{e^{\mathsf{D}}\,\gamma^D}{=} u^{\mathsf{D}}\,\gamma^D$
$\mathsf{Id}[] : (\mathsf{Id}\,a\,t\,u)[\sigma] = \mathsf{Id}\,(a[\sigma])\,(t[\sigma])\,(u[\sigma])$	$\mathsf{Id}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\hat{\Pi}\,(T : \mathsf{Set})\,(B : T \to \mathsf{Ty}\,\Gamma) : \mathsf{Ty}\,\Gamma$	$(\hat{\Pi}\,T\,B)^{\mathsf{D}}\,\gamma^D\,f$	$:\equiv (\alpha : T) \to (B\,\alpha)^{\mathsf{D}}\,\gamma^D\,(f\,\alpha)$
$(t : \mathsf{Tm}\,\Gamma\,(\hat{\Pi}\,T\,B))\,\hat{@}(\alpha : T) : \mathsf{Tm}\,\Gamma\,(B\,\alpha)$	$(t\,\hat{@}\,\alpha)^{\mathsf{D}}\,\gamma^D$	$:\equiv t^{\mathsf{D}}\,\gamma^D\,\alpha$
$\hat{\Pi}[] : (\hat{\Pi}\,T\,B)[\sigma] = \hat{\Pi}\,T\,(\lambda\alpha.(B\,\alpha)[\sigma])$	$\hat{\Pi}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\hat{@}[] : (t\,\hat{@}\,\alpha)[\sigma] = (t[\sigma])\,\hat{@}\,\alpha$	$\hat{@}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$

$$(\_)^{\mathsf{D}} : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

Displayed nat-algebras are “induction data”

$$
\begin{aligned}
\mathfrak{N}\text{-Alg} :=\, & (N \colon \mathsf{Set}) \\
& \times (z \colon N) \\
& \times (s \colon N \to N)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{N}\text{-DAlg}\ (N, z, s) :=\, & (N^{\mathsf{D}} \colon N \to \mathsf{Set}) \\
& \times (z^{\mathsf{D}} \colon N^{\mathsf{D}}\ z) \\
& \times (s^{\mathsf{D}} \colon (n \colon N) \to N^{\mathsf{D}}\ n \to N^{\mathsf{D}}(s\ n))
\end{aligned}
$$

Question: What kind of thing is the output of induction?

Sections

Fact We can define the type of **sections** of a given displayed $\mathfrak{G}$-algebra, for all GATs $\mathfrak{G}$.

Theorem Every displayed algebra over the initial $\mathfrak{G}$-algebra admits a section

- Principle of induction (e.g. $\mathbb{N}$)
- Syntax model is contextual
- Unary parametricity

$$
\begin{aligned}
\_\text{-}\mathsf{Alg} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set} \\
\mathsf{Hom} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set} \\
\_\text{-}\mathsf{DAlg} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set} \\
\_\text{-}\mathsf{Sect} &: (\mathfrak{G} : \mathtt{GAT}) \to (\Gamma : \mathfrak{G}\text{-}\mathsf{Alg}) \to \mathfrak{G}\text{-}\mathsf{DAlg}\ \Gamma \\
&\qquad \to \mathsf{Set}
\end{aligned}
$$

From [KKA19, Appendix A]

Sections

Γ^S	$: (\gamma : \Gamma^A) \to \Gamma^D\,\gamma \to \mathsf{Set}$
A^S	$: \Gamma^S\,\gamma\,\gamma^D \to (\alpha : A^A\,\gamma) \to A^D\,\gamma^D\,\alpha \to \mathsf{Set}$
σ^S	$: \Gamma^S\,\gamma\,\gamma^D \to \Delta^S\,(\sigma^A\,\gamma)\,(\sigma^D\,\gamma^D)$
t^S	$: (\gamma^S : \Gamma^S\,\gamma\,\gamma^D) \to A^S\,\gamma^S\,(t^A\,\gamma)\,(t^D\,\gamma^D)$
$\cdot^S\;\mathsf{tt}\;\mathsf{tt}$	$:\equiv \top$
$(\Gamma \triangleright A)^S\,(\gamma,\alpha)\,(\gamma^D,\alpha^D)$	$:\equiv (\gamma^S : \Gamma^S\,\gamma\,\gamma^D) \times A^S\,\gamma^S\,\alpha\,\alpha^D$
$(A[\sigma])^S\,\gamma^S\,\alpha\,\alpha^D$	$:\equiv A^S\,(\sigma^S\,\gamma^S)\,\alpha\,\alpha^D$
$\mathsf{id}^S\,\gamma^S$	$:\equiv \gamma^S$
$(\sigma \circ \delta)^S\,\gamma^S$	$:\equiv \sigma^S\,(\delta^S\,\gamma^S)$
$\epsilon^S\,\gamma^S$	$:\equiv \mathsf{tt}$
$(\sigma, t)^S\,\gamma^S$	$:\equiv (\sigma^S\,\gamma^S, t^S\,\gamma^S)$
$(\pi_1\,\sigma)^S\,\gamma^S$	$:\equiv \mathsf{proj}_1\,(\sigma^S\,\gamma^S)$
$(\pi_2\,\sigma)^S\,\gamma^S$	$:\equiv \mathsf{proj}_2\,(\sigma^S\,\gamma^S)$
$(t[\sigma])^S\,\gamma^S$	$:\equiv t^S\,(\sigma^S\,\gamma^S)$
$[\mathsf{id}]^S$	$:\equiv \mathsf{refl}$
$[\circ]^S$	$:\equiv \mathsf{refl}$
ass^S	$:\equiv \mathsf{refl}$
$\triangleright\beta_1$	$:\equiv \mathsf{refl}$
$\triangleright\beta_2{}^S$	$:\equiv \mathsf{refl}$
$\triangleright\eta^S$	$:\equiv \mathsf{refl}$
$,\circ^S$	$:\equiv \mathsf{refl}$
$\mathsf{U}^S\,\gamma^S\,T\,T^D$	$:\equiv (\alpha : T) \to T^D\,\alpha$

$(\mathsf{El}\,a)^S\,\gamma^S\,\alpha\,\alpha^D$	$:\equiv a^S\,\gamma^S\,\alpha = \alpha^D$
$\mathsf{U}[]^S$	$:\equiv \mathsf{refl}$
$\mathsf{El}[]^S$	$:\equiv \mathsf{refl}$
$(\Pi\,a\,B)^S\,\gamma^S\,f\,f^D$	$:\equiv (\alpha : a^A\,\gamma) \to B^S\,(\gamma^S, \mathsf{refl}_{a^S\,\gamma^S\,\alpha})\,(f\,\alpha)\,(f^D\,(a^S\,\gamma^S\,\alpha))$
$(\mathsf{app}\,t)^S\,(\gamma^S, \alpha^S)$	$:\equiv \mathsf{J}_{x.z.B^S\,(\gamma^S,z)\,(t^A\,\gamma\,\alpha)\,(t^D\,\gamma^D\,x)}\,(t^S\,\gamma^S\,\alpha)\,\alpha^S$
$\Pi[]^S$	$:\equiv \mathsf{refl}$
$\mathsf{app}[]^S$	$:\equiv \mathsf{refl}$
$(\mathsf{Id}\,a\,t\,u)^S\,\gamma^S\,e\,e^D$	$:\equiv \top$
$(\mathsf{reflect}\,e)^S$	$:\equiv \mathsf{UIP}$
$\mathsf{Id}[]^S$	$:\equiv \mathsf{refl}$
$(\hat{\Pi}\,T\,B)^S\,\gamma^S\,f\,f^D$	$:\equiv (\alpha : T) \to (B\,\alpha)^S\,\gamma^S\,(f\,\alpha)\,(f^D\,\alpha)$
$(t\,\hat{@}\,\alpha)^S\,\gamma^S$	$:\equiv t^S\,\gamma^S\,\alpha$
$\hat{\Pi}[]^S$	$:\equiv \mathsf{refl}$
$\hat{@}[]^S$	$:\equiv \mathsf{refl}$

$$(\_)^S : (\mathfrak{G} : \mathtt{GAT}) \to (\Gamma : \mathfrak{G}\text{-Alg}) \to \mathfrak{G}\text{-DAlg} \to \mathsf{Set}$$

$$
\begin{aligned}
\mathfrak{G}\text{-}\mathsf{Alg} &: \mathsf{Set} \\
\mathsf{Hom}_{\mathfrak{G}} &: \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set} \\
\mathfrak{G}\text{-}\mathsf{DAlg} &: \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set} \\
\mathfrak{G}\text{-}\mathsf{Sect} &: (\Gamma : \mathfrak{G}\text{-}\mathsf{Alg}) \to \mathfrak{G}\text{-}\mathsf{DAlg}\ \Gamma \\
&\qquad \to \mathsf{Set}
\end{aligned}
$$

Observation: Every GAT gives rise to a CwF of algebras and displayed algebras

	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF
Categories	$\mathfrak{Cat}$			
CwFs	$\mathfrak{CwF}$			
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	ONEGAT	

Is the **setoid model**[1] the same thing as the concrete CwF of setoids?
Is the **groupoid model** the same thing as the concrete CwF of groupoids?

[1] We mean the setoid model à la Altenkirch [Alt99], which uses **equivalence relations**, not the setoid model of Hofmann [Hof95a, Hof95b], which uses **partial equivalence relations**.

No

From [ABK+21, 7]

Displayed Setoid

$$A : \mathbf{Ty}\ \Gamma$$

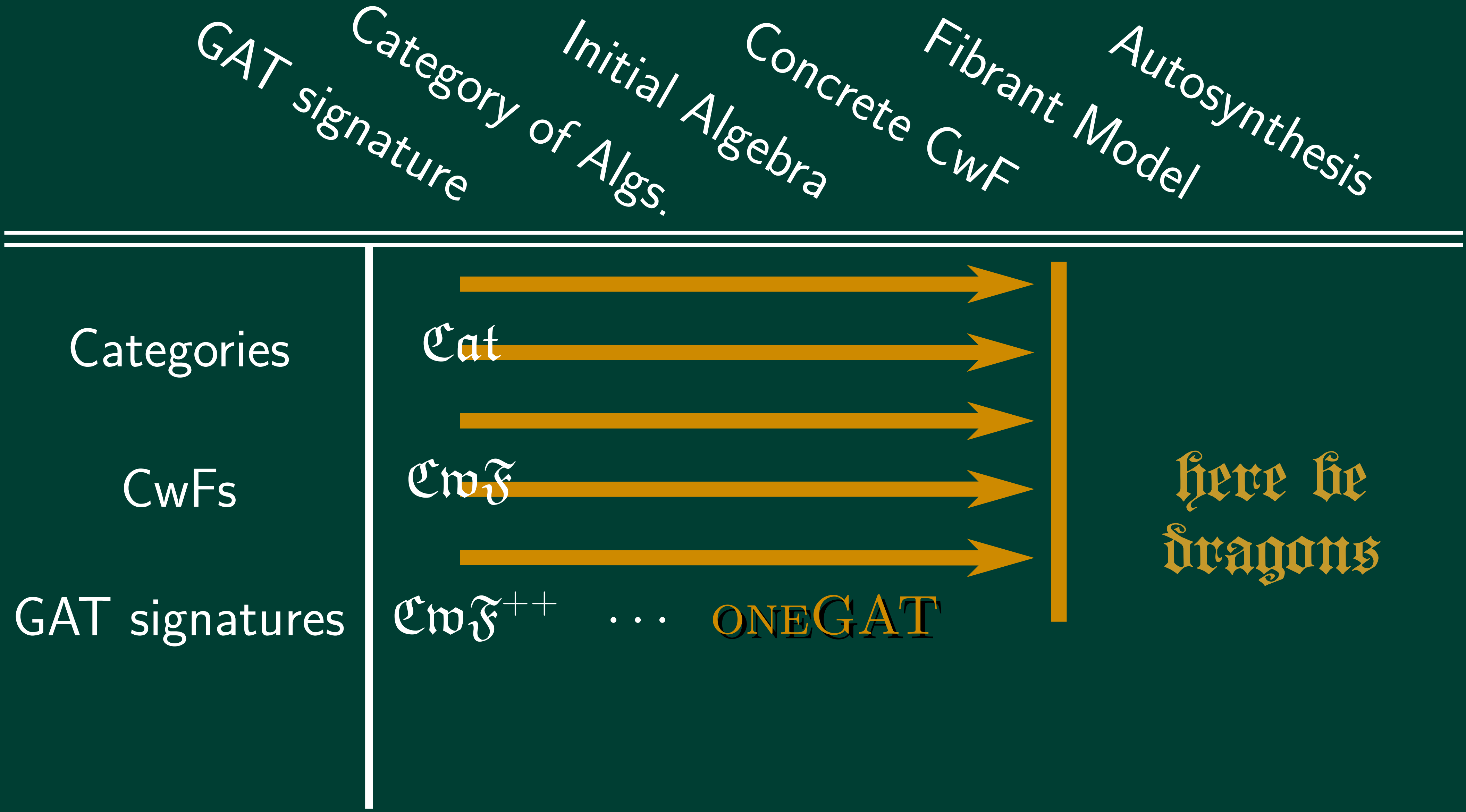
$$\begin{aligned}
|A| &: |\Gamma| \to \mathbf{Type} \\
A^{\sim} &: \{\gamma_0\ \gamma_1 : |\Gamma|\} \to \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1 \to \mathbf{Prop} \\
\mathsf{refl^*} &: \{\gamma : |\Gamma|\}(a : |A|\gamma) \to A^{\sim}\ (\mathsf{refl}\ \Gamma\ \gamma)\ a\ a \\
\mathsf{sym^*} &: \forall\{\gamma_0\,\gamma_1\,a_0\,a_1\}\{p : \Gamma^{\sim}\ \gamma_0\ \gamma_1\} \to A^{\sim}\ p\ a_0\ a_1 \to A^{\sim}\ (\mathsf{sym}\ \Gamma\ p)\ a_1\ a_0 \\
\mathsf{trans^*} &: A^{\sim}\ p_0\ a_0\ a_1 \to A^{\sim}\ p_1\ a_1\ a_2 \to A^{\sim}\ (\mathsf{trans}\,\Gamma\,p_0\,p_1)\ a_0\ a_2 \\
\mathsf{coe} &: \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1 \\
\mathsf{coh} &: (p : \Gamma^{\sim}\ \gamma_0\ \gamma_1)(a : |A|\gamma_0) \to A^{\sim}\ p\,a\,(\mathsf{coe}\,A\,p\,a)
\end{aligned}$$

Fibrancy

Idea: Carve out sub-CwFs of concrete CwFs whose types are *fibrant*

Autosynthesis

- The groupoid model provides a synthetic theory of groupoids
- The setoid model provides a synthetic theory of setoids
- The preorder model provides a synthetic theory of preorders?
- The category model provides a synthetic theory of categories?
  - Yes: [Neu25]


GAT signature
Category of Algs.
Initial Algebra
Concrete CwF
Fibrant Model
Autosynthesis
Categories
CwFs
GAT signatures
$\mathfrak{Cat}$
$\mathfrak{CwF}$
$\mathfrak{CwF}^{++}$
$\cdots$
oneGAT
here be dragons

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The End

(thank you!)