Categories with dependent and codependent arrows

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What is the categorical analogue to dependent functions?

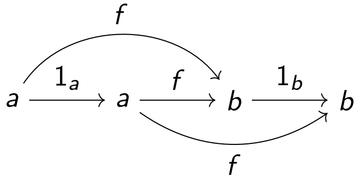
This question is different from finding categorical models for the whole of MLTT.

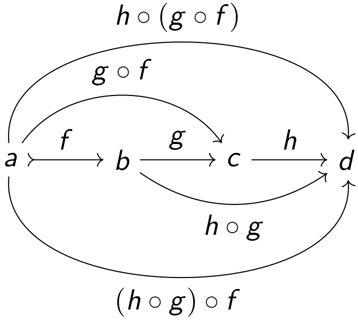
We want to model the $\prod$-type categorically

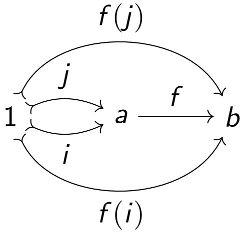
- as a fundamental notion,
- independent from a corresponding implementation of the $\sum$-type,
- and without requiring a strong background on MLTT.

How arrows generalise functions

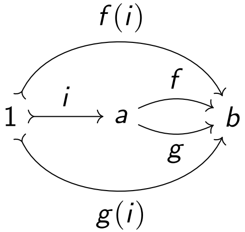
They preserve some properties of functions



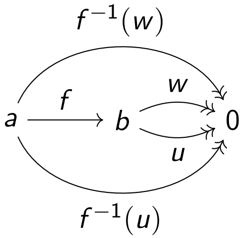


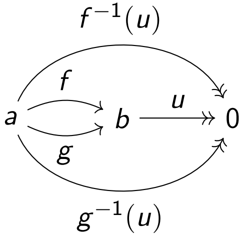


They forget some properties of functions

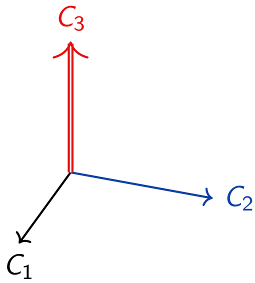


They add some properties that cannot be traced to functions



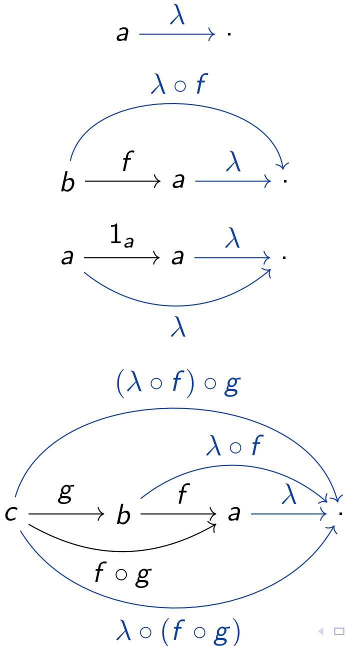


To C_1 we add family-arrows C_2 and dependent arrows C_3



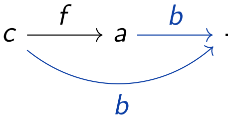
Dependent Category Theory

Categories with family-arrows $\lambda \in \mathtt{fHom}(a)$



Constant family arrows

$$a \xrightarrow{b} \cdot$$

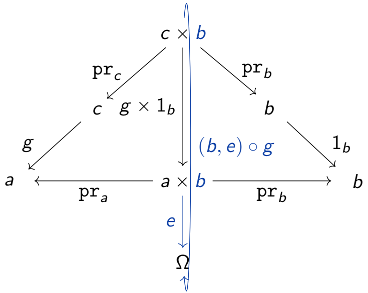


Family-arrows on a topos $\mathcal{C}$ (Pitts)

$$\mathtt{fHom}(a) := \bigcup_{b \in C_0} \mathrm{Hom}(a \times b, \Omega)$$

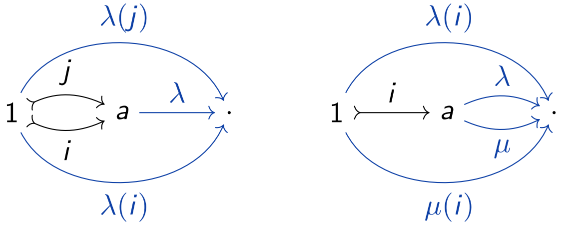
If $g\colon c \to a$, then

$$(b, e) \circ g := \big(b, e \circ (g \times 1_b)\big)$$



Fam-arrows preserve/forget properties of families of types

If $\mathcal{C}$ has 1 and $\lambda \in \mathtt{fHom}(a)$, then $i = j \Rightarrow \lambda(i) = \lambda(j)$.



$\mathcal{C}$ with 1 has the *family-arrow-extensionality property* (farExt), if

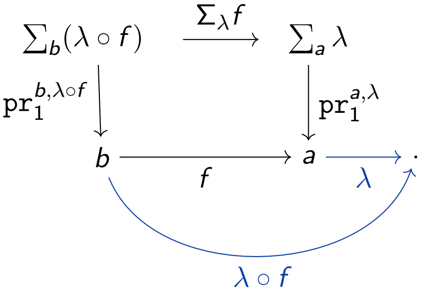
$$\forall_{i \in a}(\lambda(i) = \mu(i)) \Rightarrow \lambda = \mu$$

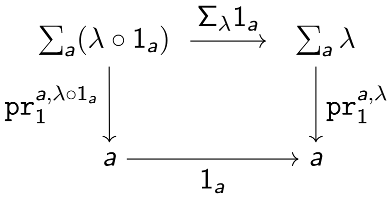
If $\mathtt{fHom}(a) := a/\mathcal{C}$, the coslice of $\mathcal{C}$ over a, and composition $\lambda \circ f$ the composition in $\mathcal{C}$, then $\mathcal{C}$ has (farExt) if and only if $\mathcal{C}$ has (arExt).

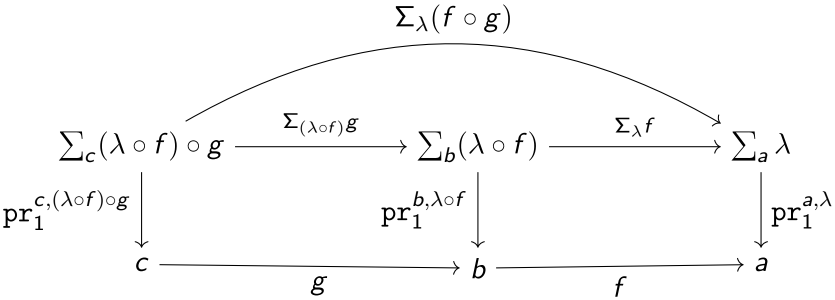
Categories with family-arrows and Sigma-objects

$$\Sigma_{\mathcal{C}} := \left(\sum_a \lambda \in C_0,\ \mathtt{pr}_1^{a,\lambda} \colon \sum_a \lambda \to a \in C_1, \right.$$

$$\left. \Sigma_\lambda f \colon \sum_b (\lambda \circ f) \to \sum_a \lambda \in C_1 \right)_{a,b \in C_0, \lambda \in \mathtt{fHom}(a), f \in \mathrm{Hom}(b,a)}$$







$(\mathtt{fam}, \Sigma)$-categories with 1 are the type-categories of Pitts (or Cartmell's categories with attributes).

If $(R, +, 0, \cdot, 1)$ is a commutative ring, and if $\mathcal{C}(R, +, 0)$ is the category of its additive, group-structure with objects a singleton $\{*\}$ and arrows the elements of R, then every commutative square

$$\begin{array}{ccc} * & \xrightarrow{a} & * \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle b} \\ * & \xrightarrow[c]{} & * \end{array}$$

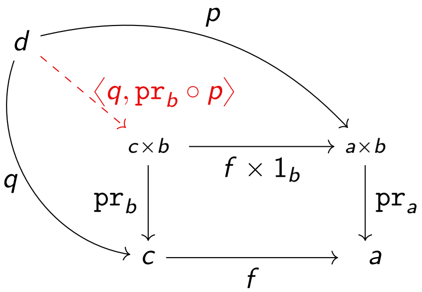
is a pullback. Let $\mathtt{Fam}(*) := R \times R$ and $(a, b) \circ c := (c + a, c + b)$. Let $\sum_*(a, b) := *$, $\mathtt{pr}_1^{*,(a,b)} := a \cdot b$, $\Sigma_{(a,b)} c := c(1 + c + b + a)$,

$$\begin{array}{ccc} * & \xrightarrow{c(1+c+b+a)} & * \\ {\scriptstyle (c+a)\cdot(c+b)}\downarrow & & \downarrow{\scriptstyle a \cdot b} \\ * & \xrightarrow[c]{} & * \end{array}$$

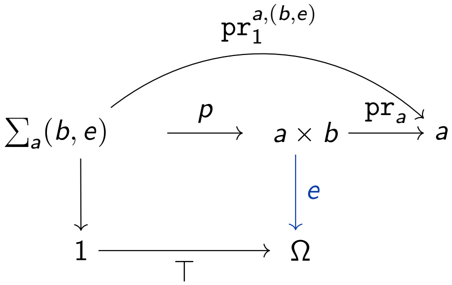
$\mathcal{C}(R, +, 0)$ is a $(\mathtt{fam}, \Sigma)$-category, which, in general, has no 1.

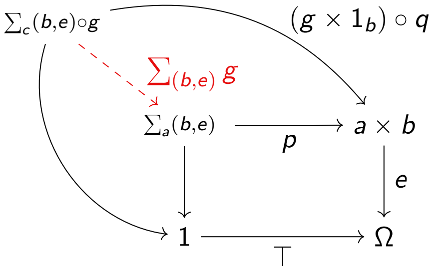
If $\mathcal{C}$ has binary products and $b \in \mathtt{fHom}(a)$,

$$\sum_a b := a \times b \ \ \& \ \ \mathtt{pr}_1^{a,b} := \mathtt{pr}_a \colon a \times b \to a.$$

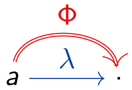


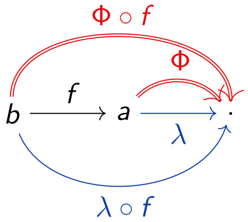
Sigma-objects on a topos (Pitts)

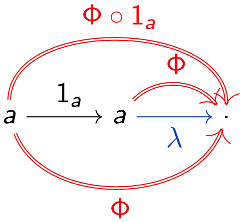




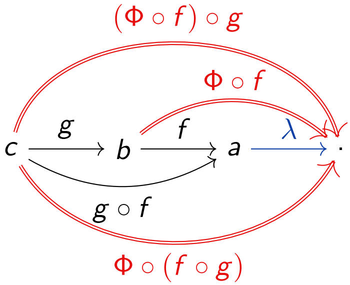
Categories with dep-arrows $\Phi \in \mathtt{dHom}(a, \lambda)$, $\lambda \in \mathtt{fHom}(a)$







$\Phi \circ 1_a$
Φ
1_a
a
a
λ
Φ



$(\Phi \circ f) \circ g$
$\Phi \circ f$
g
f
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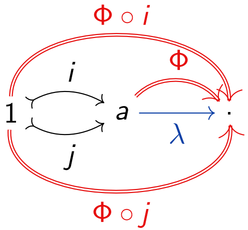
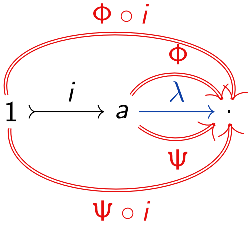
c
b
a
$g \circ f$
$\Phi \circ (f \circ g)$

Dep-arrows preserve and forget properties of dep-functions





A dep-category $\mathcal{C}$ with 1 has the dependent-arrow-extensionality property (darExt), if $\forall_{i \in a}(\Phi(i) = \Psi(i)) \Rightarrow \Phi = \Psi$

Any category $\mathcal{C}$ is turned into a dep-category I

$$\texttt{fHom}(a) := C_0$$

$$\texttt{dHom}(a, b) := \mathrm{Hom}(a, b)$$

$$f \circ g \in \texttt{dHom}(c, b \circ g) := \texttt{dHom}(c, b) := \mathrm{Hom}(c, b)$$

Any category $\mathcal{C}$ is turned into a dep-category II

$$\mathtt{fHom}(a) := S(a) := \{S_a \mid S_a \text{ is a sieve on } a\}$$

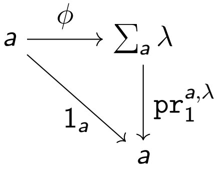
$$S_a \circ f := \{g \in \mathrm{Hom}(-, \mathrm{dom}(f)) \mid f \circ g \in S_a\}$$

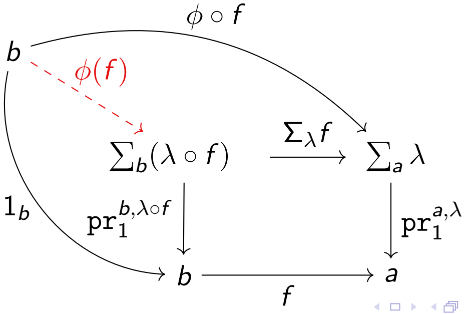
$$\mathtt{dHom}(a, S_a) := \{G(a) \mid G \text{ is a Groth top on } \mathcal{C} \text{ \& } S_a \in G(a)\}$$

$$G(a) \circ f := G(\mathrm{dom}(f)) \in \mathtt{dHom}(\mathrm{dom}(f), S_a \circ f)$$

Any $(\mathtt{fam}, \Sigma)$-category is turned into a dep-category

$$\mathtt{dHom}(a, \lambda) := \mathcal{D}_a\lambda := \left\{ \phi \in \mathrm{Hom}\left(a, \sum_a \lambda\right) \mid \mathtt{pr}_1^{a,\lambda} \circ \phi = 1_a \right\}$$





The canonical dep-structure on a topos $\mathcal{C}$

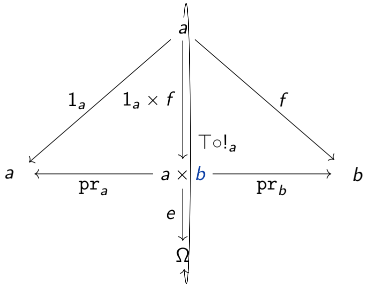
Theorem (Ehrhardt)

If $a \in \mathcal{C}$ and $(b, e) \in \mathtt{fHom}(a)$ i.e., $e\colon a \times b \to \Omega$, then

$$\mathtt{dHom}(a, (b, e)) := \left\{ \phi\colon \ \in \mathrm{Hom}\left(a, \sum_a (b, e)\right) \mid \mathtt{pr}_1^{(a,(b,e))} \circ \phi = 1_a \right\}$$

is bijective to

$$\{ f \in \mathrm{Hom}(a, b) \mid e \circ \langle 1_a, f \rangle = \top \circ !_a \}$$



There are dep-structures that are not induced by the corresponding $(\mathtt{fam}, \Sigma)$-structures

The canonical dep-structure on a commutative ring is the singleton

$$\mathtt{dHom}(*, (a, b)) := \{r \in R \mid ab + r = 0\},$$

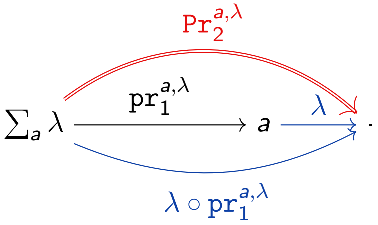
while one can define the following dep-structure

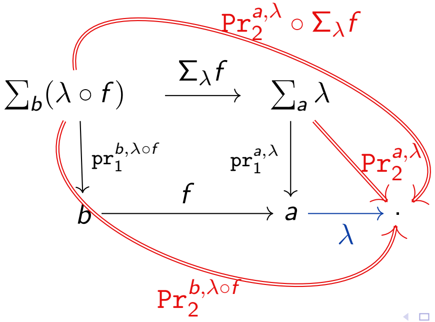
$$\mathtt{dHom}'(*, (a, b)) := \{I \in \mathtt{Ideal}(R) \mid a - b \in I\},$$

$$I \circ r := I, \quad r \in \mathrm{Hom}(*, *).$$

We can find trivially R and $a, b \in R$ with many ideals containing $a - b$.

Categories with dependent arrows and Sigma-objects

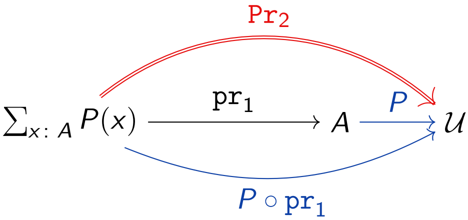




$$\mathtt{pr}_1 \colon \Big(\sum_{x\colon A} P(x)\Big) \to A, \quad \mathtt{pr}_1(a, b) := a$$

$$\mathtt{Pr}_2 \colon \prod_{z\colon \sum_{x\colon A} P(x)} P(\mathtt{pr}_1(z)), \quad \mathtt{Pr}_2(a, b) := b$$

$$z = \big(\mathtt{pr}_1(z), \mathtt{Pr}_2(z)\big)$$



If $\mathcal{C}$ has binary products and $b \in \mathtt{fHom}(a)$,

$\mathcal{C}$ is turned into a $(\mathtt{dep}, \Sigma)$-category:

$\mathtt{Pr}_2^{a,b} := \mathtt{pr}_b \in \mathtt{dHom}(a \times b, b \circ \mathtt{pr}_a) := \mathtt{dHom}(a \times b, b) := \mathrm{Hom}(a \times b, b)$,

and by the definition of $f \times 1_b$ we get

$$\mathtt{Pr}_2^{a,b} \circ \Sigma_b f := \mathtt{pr}_b \circ (f \times 1_b) = \mathtt{pr}_b =: \mathtt{Pr}_2^{c,b} = \mathtt{Pr}_2^{c,b \circ f}.$$

Theorem

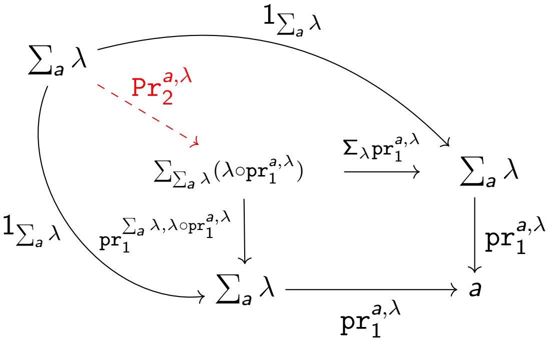
A $(\mathtt{fam}, \Sigma)$*-category* $\mathcal{C}$ *is turned into a* $(\mathtt{dep}, \Sigma)$*-category:*

$$\mathtt{Pr}_2^{a,\lambda} \in \mathcal{D}_{\sum_a \lambda}\left(\lambda \circ \mathtt{pr}_1^{a,\lambda}\right) =$$

$$\left\{\phi \in \mathrm{Hom}\left(\sum_a \lambda, \sum_{\sum_a \lambda} (\lambda \circ \mathtt{pr}_1^{a,\lambda})\right) \mid \mathtt{pr}_1^{\sum_a \lambda, \lambda \circ \mathtt{pr}_1^{a,\lambda}} \circ \mathtt{Pr}_2^{a,\lambda} = 1_{\sum_a \lambda}\right\}$$

$$\begin{tikzcd} \sum_a \lambda \arrow[r, "\mathtt{Pr}_2^{a,\lambda}"] \arrow[rr, bend right, "1_{\sum_a \lambda}"'] & \sum_{\sum_a \lambda}(\lambda \circ \mathtt{pr}_1^{a,\lambda}) \arrow[r, "\mathtt{pr}_1^{\sum_a \lambda, \lambda \circ \mathtt{pr}_1^{a,\lambda}}"] & \sum_a \lambda \end{tikzcd}$$

Proof.



$\sum_a \lambda$
$1_{\sum_a \lambda}$
$\mathtt{Pr}_2^{a,\lambda}$
$\sum_{\sum_a \lambda}(\lambda \circ \mathtt{pr}_1^{a,\lambda})$
$\Sigma_\lambda \mathtt{pr}_1^{a,\lambda}$
$\sum_a \lambda$
$1_{\sum_a \lambda}$
$\mathtt{pr}_1^{\sum_a \lambda, \lambda \circ \mathtt{pr}_1^{a,\lambda}}$
$\mathtt{pr}_1^{a,\lambda}$
$\sum_a \lambda$
a
$\mathtt{pr}_1^{a,\lambda}$

□

There are $(\mathtt{dep}, \Sigma)$-structures that are not induced by the corresponding $(\mathtt{fam}, \Sigma)$-structures

Let non-canonical dep-structure on a commutative ring

$$\mathtt{dHom}'(*, (a, b)) := \{I \in \mathtt{Ideal}(R) \mid a - b \in I\},$$

$$I \circ r := I, \quad r \in \mathrm{Hom}(*, *).$$

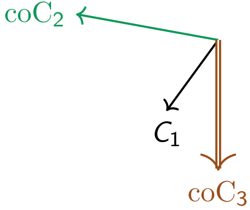
We can define

$$\mathtt{Pr}_2^{*,(a,b)} := \langle a - b \rangle \in \mathtt{dHom}'(*, (a, b) \circ ab) :=$$

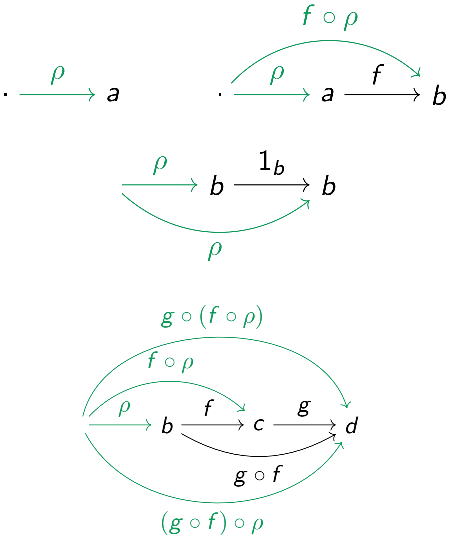
$$\mathtt{dHom}'(*, (ab + a, ab + b)) = \mathtt{dHom}'(*, (a, b)).$$

What we add that is not traced to dependent functions

To C_1 we add cofamily-arrows coC_2 and codependent arrows coC_3

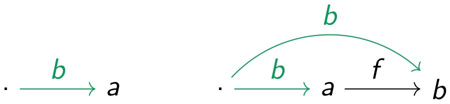


Categories with cofamily-arrows $\rho \in \texttt{cofHom}(a)$



Any category $\mathcal{C}$ is turned into a cofam-category

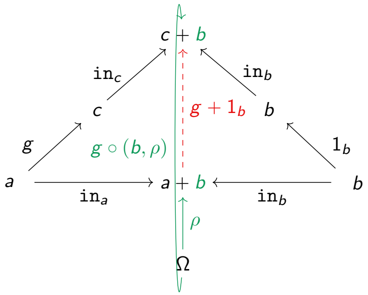
$$\texttt{cofHom}(a) := C_0$$



Cofamily-arrows on a topos $\mathcal{C}$

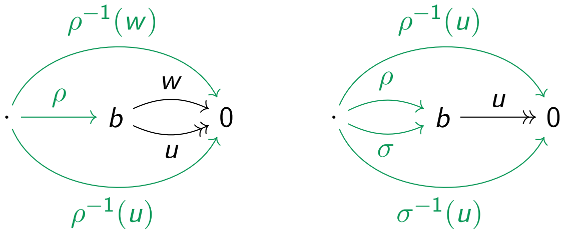
$$\texttt{cofHom}(a) := \bigcup_{b \in C_0} \mathrm{Hom}(\Omega, a + b)$$

If $g\colon a \to c$, then $g \circ (b, \rho) := \big(b, (g + 1_b) \circ \rho\big)$.



If a `cofam`-category $\mathcal{C}$ has 0, then $u = w \Rightarrow \rho^{-1}(u) = \rho^{-1}(w)$, and $\mathcal{C}$ has the cofamily-arrow-extensionality property $(\mathrm{cofarrExt})$, if

$$\forall_{u \in^{\mathrm{op}} b}(\rho^{-1}(u) = \sigma^{-1}(u)) \Rightarrow \rho = \sigma$$

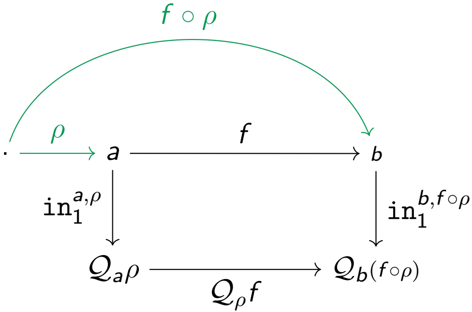


$\mathcal{C}$ becomes a `cofam`-category by taking the slices as cofamily-arrows, and then $\mathcal{C}$ has $(\mathrm{cofarrExt})$ if and only if $\mathcal{C}$ has $(\mathrm{arcoExt})$.

Categories with cofamily arrows and coSigma-objects

$$\mathcal{Q}_{\mathcal{C}} := \left(\mathcal{Q}_a\rho \in C_0,\ \mathtt{in}_1^{a,\rho}\colon a \to \mathcal{Q}_a\lambda \in C_1, \right.$$

$$\left. \mathcal{Q}_\rho f\colon \mathcal{Q}_a\rho \to \mathcal{Q}_b(f \circ \rho) \in C_1 \right)_{a,b\in C_0,\rho\in\mathtt{cofHom}(a),f\in\mathrm{Hom}(a,b)}$$

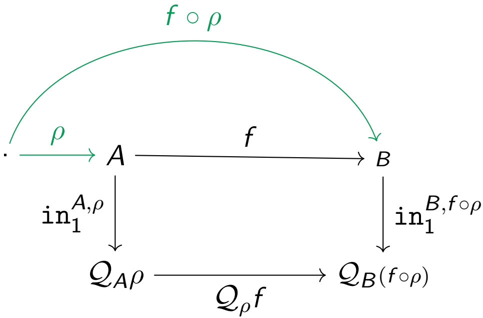


In **Set**, if $\rho\colon X \to A$ in $\mathtt{cofHom}(A)$, let

$$\mathcal{Q}_A\rho := \{\rho^{-1}(a) \mid a \in A\}$$

$$\mathtt{in}^1_{A,\rho}\colon A \to \mathcal{Q}_A\rho, \qquad a \mapsto \rho^{-1}(a)$$

$$[Q_\rho f](\rho^{-1}(a)) := (f \circ \rho)^{-1}(f(a))$$

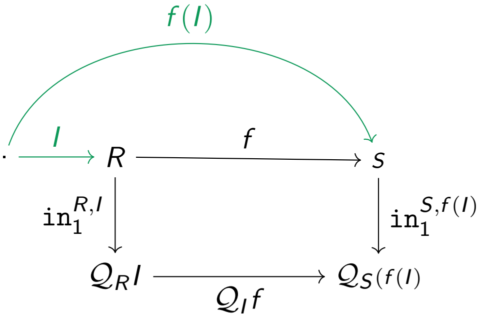


In **Ring** with arrows $f\colon R \to S$ ring-epimorphisms, if $\mathtt{cofHom}(R) := \mathcal{I}(R)$, the ideals of R, with $f \circ I := f(I)$, then

$$\mathcal{Q}_R I := R/I$$

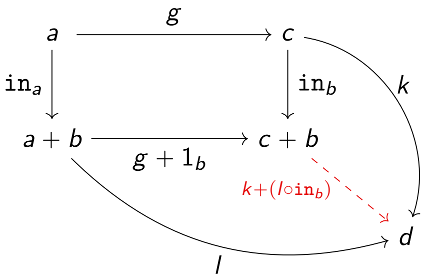
$$\mathtt{in}^1_{A,\rho}\colon A \to \mathcal{Q}_A\rho, \qquad r \mapsto r + I$$

$$\mathcal{Q}_I f\colon R/I \to S/I, \qquad [\mathcal{Q}_I f](r+I) := f(r) + f(I)$$

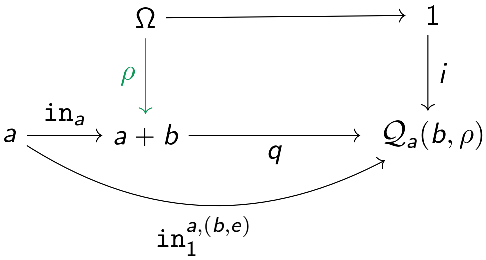


If $\mathcal{C}$ has binary coproducts and $b \in \texttt{cofHom}(a)$,

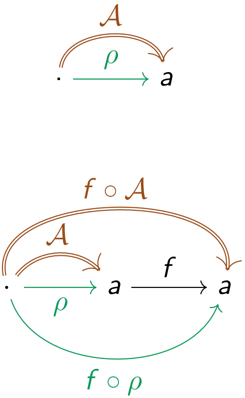
$$\mathcal{Q}_a b := a + b \quad \& \quad \texttt{in}_1^{a,b} := \texttt{in}_a \colon a \to a + b.$$

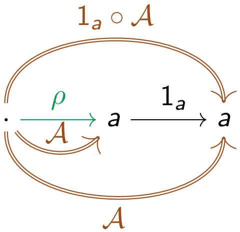


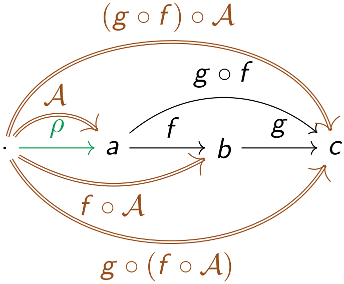
coSigma-objects on a topos



Cats with codep-arrows $\chi \in \texttt{codHom}(a, \rho), \rho \in \texttt{cofHom}(a)$




$1_a \circ \mathcal{A}$
ρ
$\mathcal{A}$
a
1_a
a
$\mathcal{A}$


$(g \circ f) \circ \mathcal{A}$
$g \circ f$
$\mathcal{A}$
ρ
a
f
b
g
c
$f \circ \mathcal{A}$
$g \circ (f \circ \mathcal{A})$

A $(\mathtt{cofam}, \mathcal{Q})$-category is a $\mathtt{codep}$-category

If $\mathcal{C}$ is a $(\mathtt{cofam}, \mathcal{Q})$-category, let for every $a \in \mathcal{C}$ and $\rho \in \mathtt{codHom}(a)$

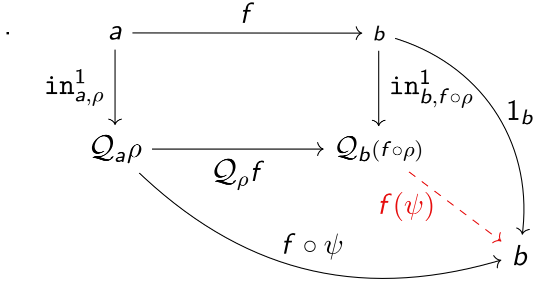
$$\mathcal{C}_a\rho := \{\psi \in \mathrm{Hom}(\mathcal{Q}_a\rho, a) \mid \psi \circ \mathtt{in}^1_{a,\rho} = 1_a\}$$

$$a \xrightarrow{\mathtt{in}^1_{a,\rho}} \mathcal{Q}_a\rho \xrightarrow{\psi} a, \qquad a \xrightarrow{1_a} a$$

be the codependent objects of ρ. If $\mathtt{codHom}(\rho, a) := \mathcal{C}_a\rho$, then $\mathcal{C}$ becomes a $\mathtt{codep}$-category.

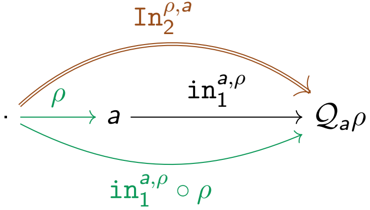
Proof:

If we write $f(\psi)$, instead of the used in the proof composition $f \circ \psi$, we get the required arrow by the universal property of pushouts.



Second injection $\mathtt{In}_2^{a,\rho}$

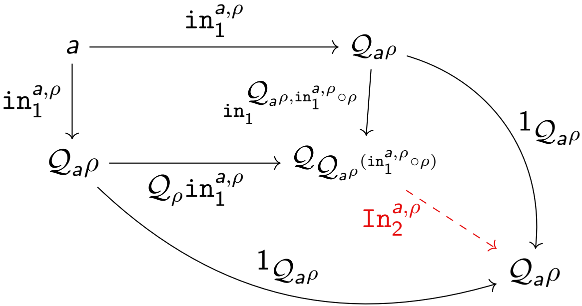
The dual to the dependent arrow $\mathtt{Pr}_2^{a,\lambda}$ is the codependent arrow $\mathtt{In}_2^{a,\rho} \in \mathtt{codHom}\left(\mathtt{in}_1^{a,\rho} \circ \rho, \mathcal{Q}_a\rho\right)$



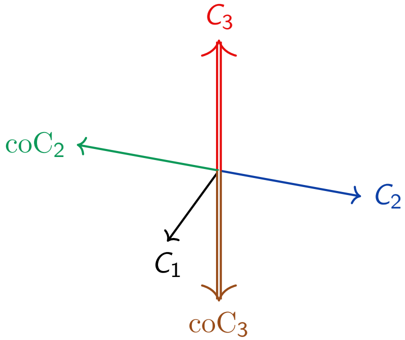
such that, for every $b \in \mathcal{C}$ and $f \in \mathrm{Hom}(a, b)$ we have that

$$\mathtt{In}_2^{b,f\circ\rho} = \mathcal{Q}_\rho f \circ \mathtt{In}_2^{a,\rho}.$$

A ($\mathtt{cofam}$, $\mathcal{Q}$)-category is a ($\mathtt{codep}$, $\mathcal{Q}$)-category



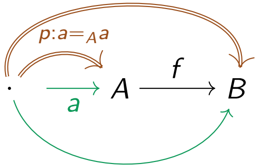
To C_1 we add C_2, C_3 and $\mathrm{coC}_2, \mathrm{coC}_3$



The category of small types $\mathcal{U}$

$$A\colon \mathcal{U}, \quad \texttt{cofam}(A) := A, \quad \texttt{codHom}(A, a) := \Omega(A, a) := a =_A a,$$

$$f \circ p := f(p)\colon \texttt{codHom}(B, f(a)) := \Omega(B, f(a)) := f(a) =_B f(a).$$

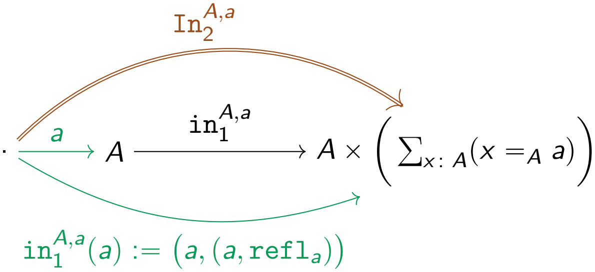


This codep-structure on $\mathcal{U}$ is not induced by the following $\mathcal{Q}$-structure on $\mathcal{U}$.

$$\mathcal{Q}_A a := A \times \left(\sum_{x \colon A} (x =_A a) \right),$$

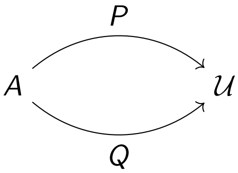
$$\mathtt{in}_1^{A,a} \colon A \to A \times \left(\sum_{x \colon A} (x =_A a) \right), \qquad a' \mapsto \left(a', (a, \mathtt{refl}_a)\right),$$

$$\mathtt{In}_2^{A,a} := \mathtt{refl}_{(a,(a,refl_a))} \colon \mathtt{codHom}\left(\mathtt{in}_1^{A,a} \circ a, \mathcal{Q}_A a\right).$$



The interplay between the dependent and codependent features of $\mathcal{U}$ is expected to lead to a good notion of type-category.

Small types form a 2-`fam`-category

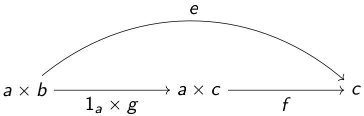


$$\mathrm{Hom}(P, Q) := \prod_{x:\,A} (P(x) \to Q(x))$$

A topos is a 2-fam-category

If (b, e) and (c, f) are in $\mathtt{fHom}(a)$, then

$$\mathrm{Hom}\big((b,e),(c,f)\big) := \{g \in \mathrm{Hom}(b,c) \mid f \circ (1_a \times g) = e\}$$



Toposes are also 2-$(\mathtt{fam}, \Sigma)$-categories, 2-$\mathtt{dep}$-categories, and 2-$(\mathtt{dep}, \Sigma)$-categories (see [9]).

Higher dep-categories

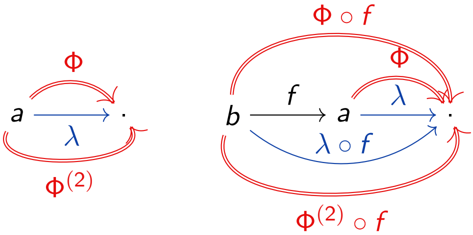
If $a \in C_0$, $\lambda \in \mathtt{fHom}(a)$, and $\Phi \in \mathtt{dHom}(a, \lambda)$, then one can define

$$\mathtt{dHom}^{(2)}\big(a, \lambda, \Phi\big),$$

such that if $\Phi^{(2)} \in \mathtt{dHom}^{(2)}\big(a, \lambda, \Phi\big)$ and $f \in \mathrm{Hom}(b, a)$, then

$$\Phi^{(2)} \circ f \in \mathtt{dHom}^{(2)}\big(b, \lambda \circ f, \Phi \circ f\big),$$

together with the obvious compatibility conditions.



In the case of $\texttt{Type}(\mathcal{U})$ a natural canidate for $\texttt{dHom}^{(2)}(A, P, \Phi)$, where $A\colon \mathcal{U}, P\colon A \to \mathcal{U}$, and $\Phi\colon \prod_{x\colon A} P(x)$ is the type of the dependent application ap_Φ of Φ i.e.,

$$\texttt{dHom}^{(2)}(A, P, \Phi) := \prod_{x,y\colon A} \prod_{p\colon x =_A y} p_*^P(\Phi_x) =_{P(y)} \Phi_y.$$

The corresponding higher Σ-objects are expected to be defined, and to behave as the dependent Σ-objects.

If $n > 2$, and $\Phi \in \mathtt{dHom}(a, \lambda), \Phi^{(2)} \in \mathtt{dHom}^{(2)}\big(a, \lambda, \Phi\big)$, ..., $\Phi^{(n)} \in \mathtt{dHom}^{(n)}\big(a, \lambda, \Phi, \ldots, \Phi^{(n-1)}\big)$, then we can define

$$\mathtt{dHom}^{(n+1)}\big(a, \lambda, \Phi, \Phi^{(2)}, \ldots, \Phi^{(n)}\big)$$

satisfying the obvious compatibility conditions with the dependent arrow-structures of lower degree. We hope to elaborate these higher dependent arrow-structures, together with their dual higher codependent arrow-structures

$$\mathtt{codHom}^{(n+1)}\big(a, \rho, \mathcal{A}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(n)}\big),$$

in future-work.

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