

On the conservativity of type theories with classical logic over arithmetic

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Conservativity

A key notion in the foundations of mathematics is the following.

Definition

Let T be a theory. We say that an extension T^+ of T is *conservative over T* if every statement expressible in the language of T and provable in T^+ , is already provable in T .

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Observation

Conservativity implies equiconsistency.

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Goal

Transfer these results to the case of classical logic – in particular replacing **HA** with Peano Arithmetic **PA**.

Classical logic in Predicative Foundations

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- ▶ Homotopy Type Theory
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If we want to obtain a classical version of Beeson's theorem, we need to replace Martin-Löf's type theory with something more appropriate...

The Minimalist Foundation

The *Minimalist Foundation* **MF** is a type theory *compatible* with the most relevant foundations of mathematics.

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For what concerns us here, **MF** can be thought of as a *predicative version* of **CIC**.

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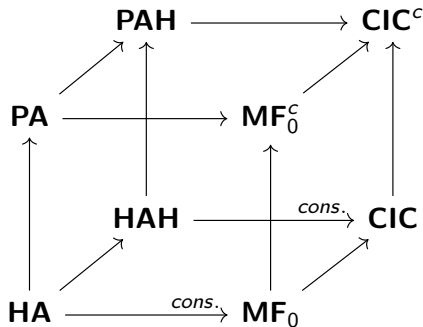
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Idea

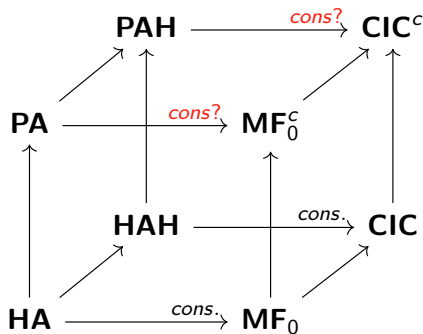
We claim that *this* result can be extended to classical logic.

A cube of theories



- ▶ x-axis (→): add type theory
- ▶ y-axis (↑): add classical logic
- ▶ z-axis (↗): add impredicativity

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- ▶ x-axis ($\rightarrow$): add type theory
- ▶ y-axis ($\uparrow$): add classical logic
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The Double-Negation Translation

If φ is an arithmetic formula, let $\varphi^{\mathcal{N}}$ be the formula obtained by prefixing a double-negation $\neg\neg$ in front of each existential quantifier and each disjunction appearing in φ .

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Theorem (Gödel, 1933)

PA $\vdash \varphi$ if and only if **HA** $\vdash \varphi^{\mathcal{N}}$.

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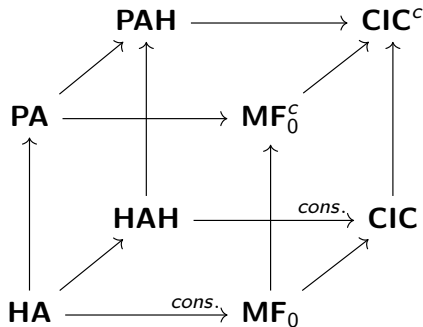
PA $\vdash \varphi$ if and only if **HA** $\vdash \varphi^{\mathcal{N}}$.

The result is readily extended to higher sorts.

Theorem (Kreisel, 1968)

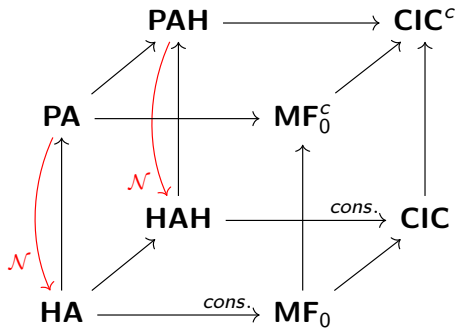
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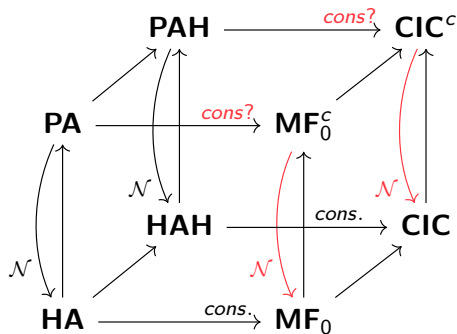
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The Challenge

In dependent type theories, logical and set-theoretical constructors are highly intertwined:

- ▶ terms appear in formulas through equality $a = b$ (as in predicate logic)
- ▶ types appear in formulas as domains of quantification $(\exists x : A)\varphi(x)$
- ▶ formulas appear in types as in the quotient set constructor A/R
- ▶ formulas appear in terms as in the subset term constructor $\{x : A \mid \varphi(x)\} : \mathcal{P}(A)$.

...we need to extend the double-negation translation to every entity!

The double-negation translation for Type Theory

In the case of **MF** and **CIC**, the definition of the translation turns out to be surprisingly simple. The relevant cases are the following.

$$\begin{aligned}(\varphi \vee \psi)^{\mathcal{N}} &\equiv \neg\neg(\varphi^{\mathcal{N}} \vee \psi^{\mathcal{N}}) \\ ((\exists x : A)\varphi)^{\mathcal{N}} &\equiv \neg\neg(\exists x : A^{\mathcal{N}})\varphi^{\mathcal{N}} \\ \mathbf{Prop}^{\mathcal{N}} &\equiv \sum_{P : \mathbf{Prop}} \neg\neg P \Rightarrow P\end{aligned}$$

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Lemma

For any type A we have that $\neg\neg \text{Eq}_{A^{\mathcal{N}}}(x, y) \Rightarrow \text{Eq}_{A^{\mathcal{N}}}(x, y)$ holds.

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Theorem (Maietti, S.)

A judgment $\mathcal{J}$ is derivable in the classical version if and only if $\mathcal{J}^{\mathcal{N}}$ is derivable in the intuitionistic version.

Conservativity over Peano Arithmetic

Theorem (Contente, S.)

MF₀^c is conservative over **PA** and **CIC** is conservative over **PAH**.

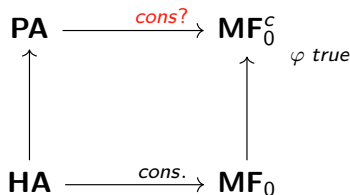
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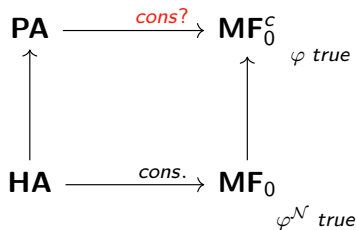
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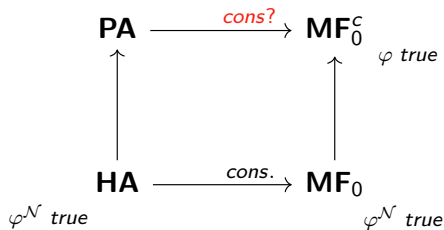
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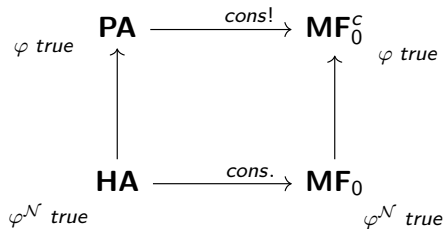
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Thanks for your attention!