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# A 2-categorical approach to the semantics of axiomatic dependent type theory

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How does the categorical approach work in axiomatic type theory?

# Intensional theory (with computation rules)

Intensional identity types

$$\frac{\vdash A : \textsc{Type}}{\begin{array}{c} x, x' : A \vdash x = x' : \textsc{Type} \\ x : A \vdash r(x) : x = x \end{array}}$$

$$\frac{\begin{array}{c} \vdash A : \textsc{Type} \\ x, x' : A;\ p : x = x' \vdash C(x, x', p) : \textsc{Type} \\ x : A \vdash q(x) : C(x, x, r(x)) \end{array}}{\begin{array}{c} x, x' : A;\ p : x = x' \vdash \mathsf{J}(q, x, x', p) : C(x, x', p) \\ x : A \vdash \qquad \mathsf{J}(q, x, x, r(x)) \equiv q(x) \end{array}}$$

Dependent sum types

$$\frac{\begin{array}{c} \vdash A : \textsc{Type} \\ x : A \vdash B(x) : \textsc{Type} \end{array}}{\begin{array}{c} \vdash \Sigma_{x:A} B(x) : \textsc{Type} \\ x : A, y : B(x) \vdash \langle x, y \rangle : \Sigma_{x:A} B(x) \end{array}}$$

$$\frac{\begin{array}{c} \vdash A : \textsc{Type} \\ x : A \vdash B(x) : \textsc{Type} \\ u : \Sigma_{x:A} B(x) \vdash C(u) : \textsc{Type} \\ x : A;\ y : B(x) \vdash c(x, y) : C(\langle x, y \rangle) \end{array}}{\begin{array}{c} u : \Sigma_{x:A} B(x) \vdash \mathsf{split}(c, u) : C(u) \\ x : A;\ y : B(x) \vdash \qquad \mathsf{split}(c, \langle x, y \rangle) \equiv c(x, y) \end{array}}$$

# *Axiomatic* theory[1] (with computation *axioms*)

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[1] Also known as *weak*, *objective*, *propositional* theory.

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# How semantics works

In a **display map category** we are given a family of display maps, denoted as $\Gamma.A \to \Gamma$ that interpret type judgements $\Gamma \vdash A : \textsc{Type}$. Term judgements $\Gamma \vdash t : A$ are interpreted as sections $\Gamma \to \Gamma.A$ of the corresponding display map.

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To formulate a model of a type constructor:

- In the **syntactic approach** one copies the type constructor into a display map category by means of choice functions in the language of the display map category.
- In the **category theoretic approach** one looks for a 1-dimensional categorical property to give to display maps that *characterises* the type constructor, allowing *a choice function as in the syntactic approach to be induced* by this property.

# How semantics works

*Example:* Identity types.

- **Syntactic approach**.
  For every display map $P_A : \Gamma.A \to \Gamma$, there is a choice of:
  - (*Form Rule*) a display map $\Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A \to \Gamma.A.A^{\blacktriangledown}$;

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- **Categorical approach.**
  *If the identity types are extensional.* For every display map $P_A : \Gamma.A \to \Gamma$, the arrow $v_A : \Gamma.A \to \Gamma.A.A^{\blacktriangledown}$ (obtained by factoring the pair $(1_{\Gamma.A}, 1_{\Gamma.A})$ through $\Gamma.A.A^{\blacktriangledown}$) is itself a display map.

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This approach can also be used for axiomatic type constructors.

**Goal.** Having a 2-dimensional structure with natural categorical conditions that allow to interpret axiomatic theory.

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$$\begin{array}{ccc} & & \Gamma.A \\ & & \downarrow \\ \Delta & \xrightarrow{f} & \Gamma \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \Delta.A[f] & \longrightarrow & \Gamma.A \\ \downarrow & \lrcorner & \downarrow \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

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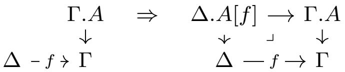


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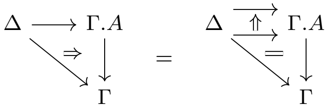

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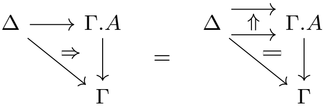


3. To have identity types with *pseudo-elimination*.

$$\begin{array}{ccc} \Omega & \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} & \Gamma.A \\ & \searrow & \downarrow \\ & & \Gamma \end{array} \quad \text{s.t.} \quad \begin{array}{c} \hom_\Gamma(\Delta, \Omega) \\ \wr\| \\ \hom_\Gamma(\Delta, \Gamma.A)^{\to} \end{array} \qquad \begin{array}{c} \Gamma.A.A'.(x = x') \\ \| \\ \Omega \end{array}$$

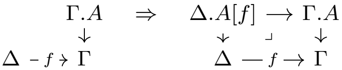
4. To have dependent sum types with *pseudo-elimination*.

$$\begin{array}{ccc} \Gamma.A.B & & \\ \downarrow & & \\ \Gamma.A & \longrightarrow & \Gamma \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \Gamma.A.B & \simeq & \Gamma.C \\ \downarrow & & \downarrow \\ \Gamma.A & \longrightarrow & \Gamma \end{array} \qquad \begin{array}{c} \Sigma_A^B \\ \| \\ C \end{array}$$

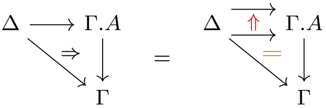
# 2-dimensional semantics of axiomatic theories

**Display map 2-categories.** (2,1)-dimensional categories with a specified class of 1-morphisms, called **display maps**, that satisfy the following conditions:

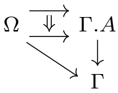
1. To substitute into types and terms.



2. To strictify eliminations in 3-4 in change of producing computation *axioms*.



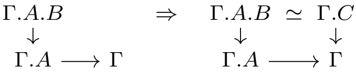
3. To have identity types with *pseudo-elimination*.



s.t.

$$\hom_\Gamma(\Delta,\Omega) \cong \hom_\Gamma(\Delta,\Gamma.A)^{\rightarrow} \qquad \Gamma.A.A'.(x=x') \doteq \Omega$$

4. To have dependent sum types with *pseudo-elimination*.



$$\Sigma_A^B \doteq C$$

# 2-dimensional semantics of axiomatic theories

**Main theorem.** *Display map 2-categories are models of axiomatic dependent type theory.*

# In detail

Under the hypotheses of the elimination rule of identity types, we are able to build a pseudo-term:

$$
\begin{array}{ccc}
\Gamma.A.A^{\blacktriangledown}.\operatorname{id}_A & \xrightarrow{\tilde{\jmath}_c} & \Gamma.A.A^{\blacktriangledown}.\operatorname{id}_A.C \\
& \varphi_A^{\Rightarrow} & \downarrow p_C \\
& = & \Gamma.A.A^{\blacktriangledown}.\operatorname{id}_A
\end{array}
$$

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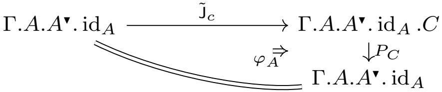
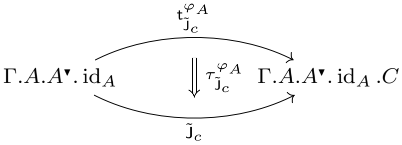



and, using the cloven isofibration structure on $P_C$, we obtain a section:

$$\mathsf{t}_{\tilde{\mathsf{J}}_c}^{\varphi_A} : \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A \to \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A\,.C$$

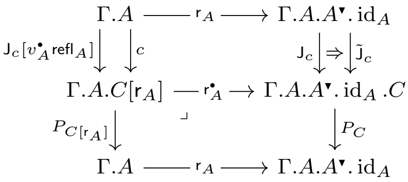
of $P_C$, at the cost of introducing an additional 2-cell:




We define $\mathsf{J}_c := \mathsf{t}_{\tilde{\mathsf{J}}_c}^{\varphi_A}$.

# In detail

Now, referring to the diagram:




we obtain a 2-cell $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A] \Rightarrow c$.

# In detail

Now, referring to the diagram:

$$\begin{array}{ccc}
\Gamma.A & \xrightarrow{\quad \mathsf{r}_A \quad} & \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A \\
{\scriptstyle \mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]}\Big\downarrow \ \Big\downarrow{\scriptstyle c} & & {\scriptstyle \mathsf{J}_c}\Big\downarrow \Rightarrow \Big\downarrow{\scriptstyle \tilde{\mathsf{J}}_c} \\
\Gamma.A.C[\mathsf{r}_A] & \xrightarrow{\quad \mathsf{r}^{\bullet}_A \quad} & \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A\,.C \\
{\scriptstyle P_{C[\mathsf{r}_A]}}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle P_C} \\
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\end{array}$$

we obtain a 2-cell $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A] \Rightarrow c$.

**Remark.** If $P_C$ is normal, then:

$$\mathsf{J}_c \mathsf{r}_A = t^{\varphi_A * \mathsf{r}_A}_{\tilde{\mathsf{J}}_c \mathsf{r}_A} = t^{1_{\mathsf{r}_A}}_{\tilde{\mathsf{J}}_c \mathsf{r}_A} = \tilde{\mathsf{J}}_c \mathsf{r}_A$$

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implying that $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]$ is in fact $c$.

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implying that $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]$ is in fact $c$.

However, if $P_C$ is just cloven, then $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]$ and $c$ can remain different.

# In detail

An application:

**A revisitation of the groupoid model.**

We consider the (2,1)-category GRPD of groupoids, functors, and natural transformations (i.e. natural isomorphisms) with **Grothendieck constructions of *pseudofunctors*** $\mathbf{\Gamma \to GRPD}$ as display maps over $\Gamma$.

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The model of axiomatic theory induced by this display map 2-category does not believe the judgemental computation rule.

*In particular, the judgemental computation rule for intensional identity type constructor is independent of the axiomatic dependent type theory.*

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Discreteness

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**Theorem.** *The display map 2-categories are precisely the models (as in the syntactic formulation) of the axiomatic theory* **extended with the discreteness rule.** *Therefore, this notion of semantics is* **sound** *w.r.t. the axiomatic theory of dependent types, and it is* **sound and complete** *w.r.t. the axiomatic theory of dependent types extended with the discreteness rule.*