Logical Predicates ~~(and Relations)~~

in Higher-order Mathematical Operational Semantics

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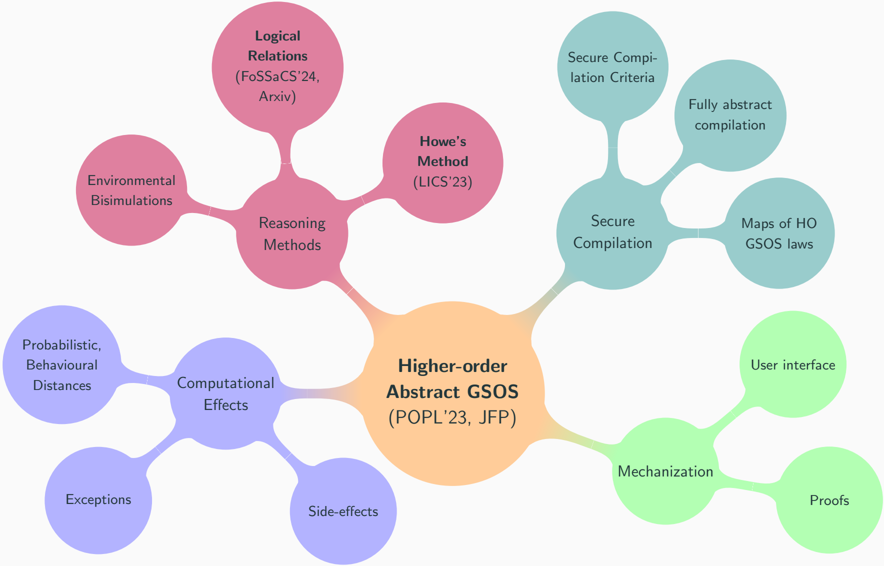
Higher-Order Mathematical Operational Semantics (or HO Abstract GSOS)



The setting of Logical Predicates

1. An operational semantics of a higher-order language is given.
   - Typically a typed λ-calculus.
   - Write $\Lambda_\tau(\Gamma)$ for the set $\{t \mid \Gamma \vdash t \colon \tau\}$ and Λ_τ for the set $\{t \mid \varnothing \vdash t \colon \tau\}$.

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4. Proceed by induction to prove that (the open extension of) $\square P$ holds.

Strong Normalization

Definition (A standard logical predicate)

$$\mathrm{SN}_{\mathsf{unit}}(t) = \Downarrow_{\mathsf{unit}}(t)$$
$$\mathrm{SN}_{\tau_1 \to \tau_2}(t) = \Downarrow_{\tau_1 \to \tau_2}(t) \wedge (\forall s \colon \tau_1 .\, \mathrm{SN}_{\tau_1}(s) \implies \mathrm{SN}_{\tau_2}(t \cdot s))$$

Definition (Open extension of SN)

$$\vec{\mathrm{SN}}_{\tau}(t)(\Gamma) = \text{For any closed substitution } (\varnothing \vdash e_n \colon \Gamma(n))_{n \in |\Gamma|}$$
$$\text{such that } \forall n \in |\Gamma| .\, \mathrm{SN}_{\Gamma(n)}(e_n), \text{ then } \mathrm{SN}_{\tau}(t[e_n/x_n])$$

Strong Normalization

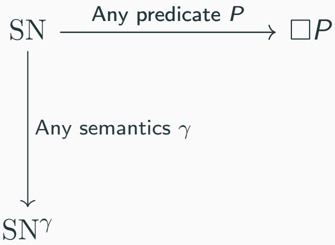
One annoying case of the proof is that of λ-abstraction $\Gamma \vdash \lambda x\colon \tau_1 . t \colon \tau_1 \rightarrow \tau_2$.
Given a substitution $(\varnothing \vdash e_n \colon \Gamma(n))_{n \in |\Gamma|}$ satisftying SN, we have to:

- Push the substitution inside the λ-abstraction, try to prove that the whole term is in SN, for that reason consider what happens when we have terms such as $(\lambda x\colon \tau_1 . t') \cdot s$ with $\mathrm{SN}_{\tau_1}(s)$ for the substituted t', think back to what happens during β-reduction, reflect on properties of substitution etc.

Complex language $\implies$ complex argument...

I will argue for two directions of abstraction, via

Higher-order Abstract GSOS



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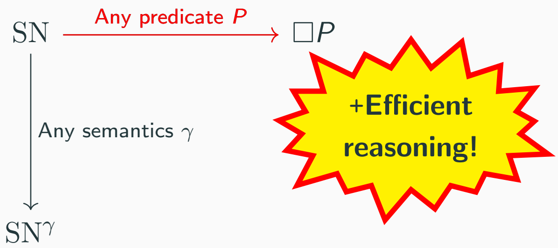

Dissecting the logical predicate (1)

$$\mathrm{SN}_{\mathsf{unit}}(t) = \Downarrow_{\mathsf{unit}}(t)$$
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$$\Downarrow\!\!\Downarrow_{\mathsf{unit}}(t) = \Downarrow_{\mathsf{unit}}(t)$$
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Idea : Abstract away from the predicate $\Downarrow$

Dissecting the logical predicate (2)

$$\Box P_{\text{unit}}\,(t) = P_{\text{unit}}\,(t)$$

$$\Box P_{\tau_1 \twoheadrightarrow \tau_2}\,(t) = P_{\tau_1 \twoheadrightarrow \tau_2}\; t \wedge (\forall s \colon \tau_1.\, t \overset{s}{\Rightarrow} t' \wedge \Box P_{\tau_1}\,(s) \implies \Box P_{\tau_2}\,(t'))$$

Dissecting the logical predicate (2)

$$\Box P_{\text{unit}}(t) = P_{\text{unit}}(t)$$

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Idea : Move one from $\Rightarrow$ to the more fundamental $\rightarrow$

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$$\Box P_{\text{unit}}(t) = P_{\text{unit}}(t)$$

greatest subset of $\Lambda_{\tau_1 \rightarrow\!\!\triangleright \tau_2}$

$$\Box P_{\tau_1 \rightarrow\!\!\triangleright \tau_2}(t) \implies P_{\tau_1 \rightarrow\!\!\triangleright \tau_2}(t) \wedge \begin{cases} \Box P_{\tau_1 \rightarrow\!\!\triangleright \tau_2}(t') & \text{if } \quad t \rightarrow t' \\ \Box P_{\tau_1}(s) \implies \Box P_{\tau_2}(t') & \text{if } \quad t \overset{s}{\rightarrow} t' \end{cases}$$

Induction up to $\boxdot$ on STLC

Theorem

Let $P \rightarrowtail \Lambda$ be any predicate on closed terms. Then P is true if all of the following are true:

1. *the unit expression* $\mathsf{e} : \mathsf{unit}$ *satisfies* P,
2. *for all closed application terms* $t\,s$ *such that* $\Box_{\tau_1 \rightarrowtriangle \tau_2} P(t)$ *and* $\Box_{\tau_1} P(s)$, *we have* $P_{\tau_2}(t\,s)$, *and*
3. *for all* λ*-abstractions* $\lambda x : \tau_1 .\, t$, *we have* $P_{\tau_1 \rightarrowtriangle \tau_2}(\lambda x : \tau_1 .\, t)$.

Proof.

Instantiate Th. 36 with $(\mathrm{Th36}.P)_\tau(\varnothing) = P_\tau$ and $(\mathrm{Th36}.P)_\tau(\Gamma \neq \varnothing) = \top$. $\Box$

Induction up to ⊡ on STLC (slightly more general)

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Let $P \rightarrowtail \Lambda$ be any predicate on closed terms. Then P is true if all of the following are true:

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3. *for all* λ*-abstractions* $\lambda x : \tau_1 .\, t : \tau_1 \to \tau_2$ *such that* $\overline{\Box P}_{\tau_2}(x : \tau_1)(t)$, *we have* $P_{\tau_1 \to \tau_2}(\lambda x : \tau_1 .\, t)$.

Proof.

Instantiate Lemma 70 (arXiv) on STLC with $(\mathrm{Lem70}.P)_\tau(\varnothing) = Q_\tau$ and $(\mathrm{Lem70}.P)_\tau(\Gamma \neq \varnothing) = \top$. □

Let's try this out!

Proving strong normalization for STLC

1. $\Downarrow_{\mathsf{unit}}(\mathsf{e})$;
2. $\Downarrow_{\tau_2}(t\,s)$ with $\Box_{\tau_1 \rightarrow \tau_2} \Downarrow (t)$ and $\Box_{\tau_1} \Downarrow (s)$;
3. $\Downarrow_{\tau_1 \rightarrow \tau_2}(\lambda x \colon \tau_1.\, t)$ (what t can do is irrelevant in this case).

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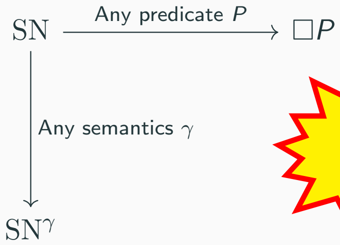
Proof.

(1) and (3) are trivial, (2) is straightforward once you realize that $\Box Q$ is an **invariant** w.r.t. $\rightarrow$ for all Q. $\Box$

Objective Complete

Let's explore the other direction



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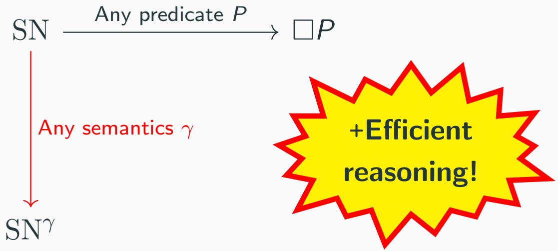

The abstract setting of Logical Predicates

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Categorical machinery

asdf

$$B(X,Y) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C} \quad \gamma \colon \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$$

$$B(X,Y) = Y + Y^X \qquad \gamma(t) = t' \text{ if } t \to t' \text{ and } \gamma(\lambda x.M) = (e \mapsto M[e/x])$$

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\mathbf{Pred}(\mathcal{C})^{\mathsf{op}} \times \mathbf{Pred}(\mathcal{C}) & \xrightarrow{\overline{B}} & \mathbf{Pred}(\mathcal{C}) \\
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\end{array}$$

For example, $\overline{B}(P,Q) \subseteq \mu\Sigma + \mu\Sigma^{\mu\Sigma}$ is the disjoint union of (i) the set $\{t \mid Q(t)\}$ and (ii) the set of functions $f \in \mu\Sigma^{\mu\Sigma}$ that map inputs in P to outputs in Q.

Logical Predicates

Relative invariant

Let $c\colon Y \to B(X, Y)$ be a $B(X, -)$-coalgebra. Given predicates $S \rightarrowtail X$, $P \rightarrowtail Y$, we say that P is an S-relative ($\overline{B}$-)invariant (for c) if

$$P \leq c^{\star}[\overline{B}(S, P)].$$

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1. If $t \to t'$, then $P(t')$ (with ND: if $\exists t.\, t \to t'$, then $P(t')$).
2. For all s, if $t \xrightarrow{s} t'$ and $P(s)$, then $P(t')$.

One logical predicate to rule them all

The $\Box$

Under certain conditions, the most important being that the predicate lifting $\overline{B}$ is **predicate-contractive**, for every predicate $P \rightarrowtail X$ on the state space of our coalgebra $X \to B(X, X)$ (i.e. a program property), there exists a certain "large" predicate $\Box P$ such that:

1. $\Box P \leq P$
2. $\Box P \leq c^{\star}[\overline{B}(\Box P, \Box P)]$ (i.e. $\Box P$ is logical)
3. $\Box P$ is the largest $\Box P$-relative invariant.

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3. $\Box P$ is the largest $\Box P$-relative invariant.

Conclusion/translation: The lifting being defined inductively on types is sufficient for the existence of this magical, suitable logical predicate.

Logical Predicates proof method in the abstract

Assuming the following:

1. An initial algebra (object of terms) $\Sigma\mu\Sigma \xrightarrow{\iota} \mu\Sigma$,
2. an "operational semantics" morphism $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ for some bifunctor $B \colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$,
3. and logical predicates $\square(-)$,

the proof method of logical predicates amount to the following:

Fundamental Property

As initial algebras have no proper subalgebras, then

$$\overline{\Sigma}(\square P) \leq \iota^{\star}[\square P] \implies \square P \cong \mu\Sigma \implies P \cong \mu\Sigma.$$

Induction up to $\Box$

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Induction up to $\boxdot$

For a certain class of λ-**laws**, instead of laboriously showing $\overline{\Sigma}(\boxdot P) \leq \iota^\star[\boxdot P]$, it suffices to show the much simpler $\overline{\Sigma}(\Box P) \leq \iota^\star[P]$.

Thank you!