Controlling unfolding in type theory

Daniel Gratzer1Jonathan Sterling1Carlo Angiuli2Thierry Coquand3Lars Birkedal1WG6 Meeting2023-04-23

Aarhus University, Carnegie Mellon University, Chalmers University

What differentiates a core theory from an actual proof assistant?

- Advanced features: implicit arguments, unification, pattern-matching
- Intermediate features: termination checking, schemata for inductive types
- Very basic features: definitions

Our goal: improve the UX of a feature by pushing the core theory to include it.

Turns out this is hard, so let's start with the basics: definitions Crucial point:

two : ℕ two ≜ 2

 $_{-}$: two = 2 $_{-} \triangleq refl$

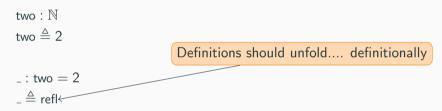
Definitions in proof assistants

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Hardly a startling insight, but it is rather crucial; only way to prove something

Fully translucent definitions certainly work, but not without cost.

Pros of unfolding	Cons of unfolding	
We can prove things	Goals become unreadable	
	Type-checking performance degrades	
	Increases coupling between implementation and use	

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In practice, the left-hand column wins.

We can't just refuse to unfold definitions, but we can control when it happens...

- Default opaque/abstract definitions
- Users may explicitly unfold a definition within a fixed scope
- The system tracks dependencies to ensure type-soundness
- Unfolding should be silent in terms; can't obstruct further computation

Library authors leave things abstract-by-default. If a user must unfold, they can.

Our core idea is to design a mechanism satisfying these desiderata

- We revisit the type-theoretic account of translucent definitions (singleton types)
- Refine this idea by replacing singleton types with extension types
- Show that extension types can be used to encode semi-translucent definitions
- Propose a surface syntax/elaboration mechanism

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Starting with the core language makes it easy to propose various extensions Interesting type theory to be found even in this most basic feature. How does one express translucent definitions type-theoretically?

- Each definition will be encoded by a variable
- ... but with a fancy type.
- This idea doesn't come from dependent type theory, but from module systems

Encode a definition $x : A \triangleq M$ through a type containing only one element: M.

For a given type M : A, we define the singleton type $S_A(M)$ by the following rules:

$$\frac{N:A}{N:S_A(M)} \qquad \qquad \frac{N:S_A(M)}{N:A} \qquad \qquad \frac{N:S_A(M)}{N=M:A}$$

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Hypothesizing over a variable $x : S_A(M) \iff$ working relative to $x : A \triangleq M$

Very roughly, we have the following:

• Opaque definitions:

$$x : A \triangleq M \iff x : A \cong \sum_{a:A} \bot \to (a = M)$$

• Translucent definitions:

$$x: A \triangleq M \iff x: S_A(M) \cong \sum_{a:A} \top \to (a = M)$$

Either we never gain access to the proof a = M or we're always stuck with it.

Translucent definitions versus abstract definitions

Very roughly, we have the following:

• Opaque definitions:

$$x: A \triangleq M \iff x: A \cong \sum_{a:A} \bot \to (a \stackrel{\checkmark}{=} M)$$

(For the sake of this slide: extensional equality)

• Translucent definitions:

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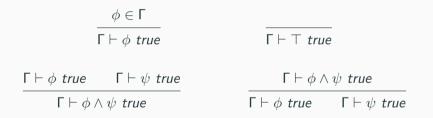
- Key idea: let's allow propositions other than \top and \bot .
- We need a universe of very strict propositions $\mathbb{F}.$
- Close \mathbb{F} under (at least) \top and \wedge .

Notation and properties inspired by cofibrations from cubical type theory.

(Spoilers): \mathbb{F} isolates subshapes $\rightsquigarrow \mathbb{F}$ classifies which definitions unfold.

New form of context: Γ , ϕ .

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New form of judgment \Gamma \vdash \phi true:
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"Very strict": user never has to write proofs for elements of $\mathbb{F}.$

 $\frac{\phi: A \text{ type}}{\phi \to A \text{ type}}$

$\phi \vdash \pmb{M} : \pmb{A}$	$M:\phi ightarrow A$	ϕ true
$\overline{\langle \phi \rangle M : \phi \to A}$	M! : A	

Two new type formers: partial element types

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$$\frac{\phi: A \text{ type}}{\phi \to A \text{ type}}$$

$$\frac{\phi \vdash M: A}{\phi M: \phi \to A}$$

$$\frac{M: \phi \to A \quad \phi \text{ true}}{M!: A}$$

$$\frac{A \text{ type } \phi \vdash M : A}{\{A \mid \phi \hookrightarrow M\} \text{ type}}$$
$$\frac{N : A \quad \phi \vdash N = M : A}{\text{in}(N) : \{A \mid \phi \hookrightarrow M\}} \qquad \qquad \frac{N : \{A \mid \phi \hookrightarrow M\}}{\text{out}(N) : A}$$

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$$\frac{N: \{A \mid \phi \hookrightarrow M\} \quad \phi \text{ true}}{\operatorname{out}(N) = M: A}$$

Two new type formers: extension types

$$\begin{array}{c} A \text{ type } \phi \vdash M : A \\ \hline \{A \mid \phi \hookrightarrow M\} \text{ type} \end{array}$$

$$\begin{array}{c} N : A \\ \hline in(N) : \{A \mid \phi \hookrightarrow M\} \end{array}$$

$$\begin{array}{c} N : \{A \mid \phi \hookrightarrow M\} \\ \hline out(N) : A \end{array}$$

$$\frac{N: \{A \mid \phi \hookrightarrow M\} \quad \phi \text{ true}}{\operatorname{out}(N) = M: A}$$

We can make good on an earlier promise:

$$S_A(M) = \{A \mid \top \hookrightarrow M\}$$

op is always true, so

$$\frac{N:S_A(M)}{\mathsf{out}(N)=M:A}$$

We haven't added \bot , but if we did we could prove $\{A \mid \bot \hookrightarrow M\} \cong A$

Fix a definition $x : A \triangleq M$.

- 1. Associate a fresh proposition symbol Υ_{x} to the definition.
- 2. Encode the definition as a constant $x : \{A \mid \Upsilon_x \hookrightarrow M\}$.
- 3. Replace subsequent occurrences of x with out(x).

Taking $\Upsilon_{\chi} = \top$ gives normal definitions.

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Taking $\Upsilon_{\chi} = \top$ gives normal definitions.

If Υ_x is some fresh symbol, how can we ever unfold this definition?

Short answer: more extension types.

- We first consider how to unfold definitions for an entire subsequent definition.
- In our above language, dictionary, have

$$x: \{A \mid \Upsilon_x \hookrightarrow M\} \qquad y: \{B \mid \Upsilon_y \hookrightarrow N\}$$

• If we want to make sure x unfolds definitionally in N, force $\Upsilon_y \implies \Upsilon_x$

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We check N after assuming Υ_y \implies so Υ_x holds when checking N \implies so out(x) = M in N

This is why we want to be sure to check N as a partial element!

Big idea II

Fix a definition $x : A \triangleq M$.

- 1. Specify which definitions x unfolds e.g. $y_0 \ldots y_n$
- 2. Associate a fresh proposition symbol Υ_x to the definition.
- 3. Add the following principle:

 $\frac{\Gamma \vdash \Upsilon_x \ true}{\Gamma \vdash \Upsilon_{y_i} \ true}$

- 4. Encode the definition as a constant $x : \{A \mid \Upsilon_x \hookrightarrow M\}$.
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Warning

A bunch of ways to specify what it means to add these propositions/inequalities.

Don't worry about it.

- Normally, a program is a sequence of definitions
- For us then, a program is a sequence of axioms
- Each axiom either specified a proposition, an inequality, and an extension type.

 $\begin{array}{l} \operatorname{prop} \Upsilon_{\operatorname{neg}} \\ \operatorname{axiom} \operatorname{neg} : \mathbb{Z} \to \mathbb{Z} \\ \operatorname{neg} \triangleq \dots \\ \operatorname{invol} : (n : \mathbb{Z}) \to \operatorname{neg}(\operatorname{neg} n) = n \\ \operatorname{invol} \triangleq \dots \end{array} \xrightarrow{} \operatorname{prop} \Upsilon_{\operatorname{invol}} \\ \begin{array}{l} \underset{\operatorname{requality}}{\operatorname{axiom}} \operatorname{neg} : \{\mathbb{Z} \to \mathbb{Z} \mid \Upsilon_{\operatorname{neg}} \hookrightarrow \dots\} \\ \underset{\operatorname{inequality}}{\operatorname{orm}} \Upsilon_{\operatorname{invol}} \leq \Upsilon_{\operatorname{neg}} \\ \operatorname{axiom} \operatorname{invol} : \\ \\ \{(n : \mathbb{Z}) \to \operatorname{neg}(\operatorname{neg} n) = n \mid \Upsilon_{\operatorname{comm}} \hookrightarrow \dots\} \end{array}$

This is the beginning of some informal elaboration strategy

- Automatically "type-safe"
- Automatically invariant under conversion (replacing equals by equals)
- Equations are *definitional* and don't produce coherence hell!

One nice example of how this methodology helps:

 ${\bf Q}.$ Does unfolding A in B allow this unfolding in the type of B?

A. No! Extension types require the type to be fully defined!

Crucial point, otherwise uses might be ill-formed!

This translation surfaces two ways to use a prior definition:

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Just caring about the type; default usage

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Caring about every single aspect of the definition; occasionally necessary

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- Opaque usage
- Transparent usage

Crystallized by whether we require $\Upsilon_x \leq \Upsilon_y$.

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- If A \rightarrow B is transparent and B \rightarrow C is transparent, so is A \rightarrow C.
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Necessary for "subject reduction".

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 Solved through elaboration!
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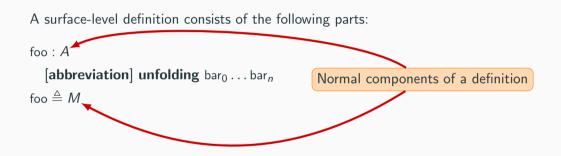
- Now that we have a target core language in place, we want nice syntax
- Should abstract a bit, but the translation should be simple and predictable
- In particular, the transformation should be compositional and local

We will define the surface syntax by elaboration.

- No typing judgments per se, just elaboration judgments
- Tautologically, well-formed surface programs produce well-formed core terms

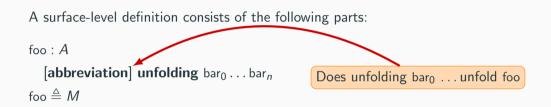
A surface-level definition consists of the following parts:

foo : A[abbreviation] unfolding bar₀... bar_n foo $\triangleq M$



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foo : A[abbreviation] unfolding bar₀... bar_n What is unfolded in Mfoo $\triangleq M$



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```
foo : A
[abbreviation] unfolding bar<sub>0</sub>... bar<sub>n</sub>
foo \triangleq M
```

M may make use definitions other than bar_i! They just won't unfold

Most of this is familiar, except **abbreviation**.

Almost identical, except:

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Returning to 3-for-2, this gives us one of the two outstanding implications.

- Many, many convenience features are possible.
- We'll settle for one: local unfolds

TLDR: a construct to create a local scope where a definition unfolds.

What if we do want something to unfold in a type?

- Obvious issue: could this be used without this unfolding?
- Potentially yes...
- ... provided no details of the type were exposed

Just create an auxiliary definition for the type which unfolds things.

two $\triangleq 2$

$\begin{array}{l} {\tt tp}: \mathcal{U} \text{ abbreviation unfolding two} \\ {\tt tp} \triangleq (p: {\tt two} = 2) \rightarrow p = {\tt refl} \end{array}$

contr : tp $contr \triangleq ...$

- If two isn't unfoldable, well-formed but useless.
- If two is unfoldable, vanishes definitionally.

The basic idea: a new expression form

unfold foo in M

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unfold foo in ${\cal M}$

two $\triangleq 2$

contr : **unfold** two **in** $(p : two = 2) \rightarrow p = refl contr \triangleq \dots$

A few complications

- What should this expression be equal to?
- What about the type of *M*?
- Type may not even be well-formed without some unfolding...

Local unfold through elaboration

Roughly, we elaborate local unfolds by hoisting:

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To elaborate

foo : B unfolding $bar_0 \dots bar_n$ foo $\triangleq N(unfold bar in M)$

Will produce/use the following:

$$\textbf{axiom} \text{ hoisted}: \Upsilon_{\mathsf{bar}_0} \to \dots \to \Upsilon_{\mathsf{bar}_n} \to \{A \mid \Upsilon_{\mathsf{bar}} \hookrightarrow M\}$$

Replace **unfold** bar **in** M with out(hoisted)! · · · !

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Local unfolds need partial element types!

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Still easy to reason about: just encoding a common design pattern.

How can we actually crystallize this?

- Define several *elaboration judgments*
- Term-level components look like fancy bidirectional type-checking
- \bullet Should be decidable \leadsto elaboration can be implemented

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- Define several *elaboration judgments*
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- \bullet Should be decidable \leadsto elaboration can be implemented

Decidable iff conversion in the core language is decidable, so normalization

Output of elaboration must be a *signature* in the core language

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Not explained: signatures induce a context ("A is well-formed wrt Σ ").

Elaboration is controlled by 4 key judgments:

$$\Sigma \vdash \vec{S} \rightsquigarrow \Sigma'$$

$$\Sigma; \Gamma \vdash \tau \Leftarrow type \rightsquigarrow \Sigma', A$$

$$\Sigma; \Gamma \vdash e \Leftarrow A \rightsquigarrow \Sigma', M$$

$$\Sigma; \Gamma \vdash e \Rightarrow A \rightsquigarrow \Sigma', M$$

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The judgments for elaboration

Elaboration is controlled by 4 key judgments:

 $\Sigma \vdash \vec{S} \rightsquigarrow \Sigma'$ $\checkmark \Sigma; \Gamma \vdash \tau \Leftarrow type \rightsquigarrow \Sigma', A$ $\Sigma: \Gamma \vdash \mathbf{e} \Leftarrow A \rightsquigarrow \Sigma'. M$ Elaborate a type; Σ : input signature $\Sigma: \Gamma \vdash \mathbf{e} \Rightarrow A \rightsquigarrow \Sigma'. M$ **F**: local variables hoist local-unfolds into Σ' Invariant: A wf wrt Σ, Γ, Σ'

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Elaborate a term
A is given & wf'd
Output is a core term

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Bidirectionalism minimizes user-provided annotations.

.

$$\frac{\Sigma; \Gamma \vdash e_0 \Rightarrow (x : A) \rightarrow B(x) \rightsquigarrow \Sigma_1; M}{\Sigma; \Gamma \vdash e_1 \Leftarrow A \rightsquigarrow \Sigma_2; N}$$
$$\frac{\Sigma; \Gamma \vdash e_0(e_1) \Rightarrow B[N/x] \rightsquigarrow \Sigma_1, \Sigma_2; M(N)$$

- Elaborate e_0 , get the type $(x : A) \rightarrow B(x)$ along with the M
- Elaborate e_1 using the type we just computed from e_0
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(Mostly to convince you that someone considered this)

$$\Sigma; \Gamma, \Upsilon_{\vartheta} \vdash \mathbf{e} \Leftarrow A \rightsquigarrow \Sigma_{1}; M$$

$$\mathbf{let} \ \chi := gensym ()$$

$$\mathbf{let} \ \Sigma_{2} := \Sigma_{1}, \mathbf{axiom} \ \chi : \prod_{\Gamma} \{A \mid \Upsilon_{\vartheta} \hookrightarrow M\}$$

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- Close up M; extend Σ_1 with hoisted-out constant.
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- Recursively elaborate e, get some core term M and type A
- Ensure the term we're checking against matches the synthesized type

$$\begin{split} & \Sigma; \Gamma \vdash \mathbf{e} \Rightarrow A \rightsquigarrow \Sigma_1; M \\ & \frac{\Gamma \vdash A = B \text{ type}}{\Sigma; \Gamma \vdash \mathbf{e} \Rightarrow A \rightsquigarrow \Sigma_1; M} \end{split}$$

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One final foray into some theory.

- As indicated before, elaboration should be decidable.
- So we need to decide conversion in the core theory.
- Our approach: normalization
- Our approach to this approach: Synthetic Tait Computability

The hard bit: the conditional rule for extension types

- Crucial step in normalization proofs: carve out renamings
- Big problem: the neutrality of **out**(e) isn't stable under renamings

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- Authors 2 & 3 already considered STC for cubical type theory (similar problems)
- Reuse a key idea: unstable neutrals

TLDR: type theory with extension types & partial element types enjoys normalization.

Further details are banished to bonus slides.

Currently, there are two implementations of controlled unfolding:

- cooltt: already had extension types, implemented as described above. https://github.com/RedPRL/cooltt
- Agda: doesn't use extension types, implemented by Amélia Liao & Jesper Cockx https://github.com/agda/agda/pull/6354

(Interested in adding controlled unfolding to your proof assistant? I'm around.)

We can implement controlled unfolding without fancy types, so why bother with them?

- To structure the proof of decidability of conversion
- To guide us in various design choices (what is unfolded where)
- Give a reference for users to reason about to predict interactions

However, don't have to implement extension types to use controlled unfolding!

A few interesting questions remain...

- What's the best way for this to interact with unification?
- Can we describe unfolding recursive definitions only a fixed number of times?
- What about data types? Can we interpolate between Σ 's and records?
- What other features of proof assistants benefit from this attention?

- We revisit the type-theoretic account of translucent definitions (singleton types)
- Refine this idea by replacing singleton types with extension types
- Show that extension types can be used to encode semi-translucent definitions
- Propose a surface syntax/elaboration mechanism

- Work internally to a presheaf topos to define the normalization model
- Each type former is modeled in turn, as a sequence of programming exercises.
- Each type is equipped with reify/reflect operations.
- $\bullet\,$ Used for cubical type theory, multimodal type theory, and $\infty\text{-type}$ theories.

Cubical type theory is the most relevant: it also has extension types.

The proof of normalization is almost standard, except for aforementioned issue.

- Standard normalization uses normals and neutrals
- We can't have neutrals, but we can have neutrals keyed by a proposition
- Big idea: proposition represents when the neutral isn't meaningful

Key case: the neutral for out_{ϕ} is associated to ϕ .

Reflect function becomes more complicated:

reflect : $(M : \operatorname{Tm}(A))(\phi : \mathbb{P}) (e : \operatorname{Ne} A \phi M) (M^{\bullet} : \phi \to \operatorname{Tm}^{\bullet}(A, M))$ $\to {\operatorname{Tm}^{\bullet}(A, M) | \phi \hookrightarrow M^{\bullet}}$

Informally: you just provide the answer when the neutral doesn't help.