Higher Inductive Types in Coinductive Definitions via Guarded Recursion.

Magnus Baunsgaard Kristensen **Rasmus Ejlers Møgelberg**
Andrea Vezzosi

IT University of Copenhagen

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Coinduction

- Streams

$$\mathsf{Str}(A) \simeq A \times \mathsf{Str}(A)$$

- Productive stream definitions

$$\text{zeros} := 0 :: \text{zeros}$$
$$\text{map}\, f\, (x :: xs) := f(x) :: (\text{map}\, f\, xs)$$

- Non-productive stream definitions

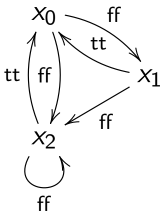
$$\text{undef} := 0 :: \text{tl}(\text{undef})$$

- Encoding productivity
  - Syntactic checks
  - Sized types
  - Guarded recursion
- Working directly with corecursion (Paco)

Coinduction

- Much previous work on M-types
- But how about non-deterministic processes?

$$\mathrm{LTS} \simeq \mathrm{P_f}(A \times \mathrm{LTS})$$

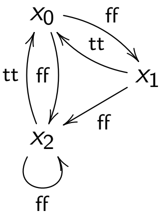


- Or probabilistic processes?

Coinduction in type theory

- How to represent functors like $\mathsf{P_f}$?
- How to recursively define *productive* processes?

$$
\begin{aligned}
\mathsf{LTS} &\simeq \mathsf{P_f}(A \times \mathsf{LTS}) \\
x_0 &= \mathsf{fold}(\{(\mathsf{ff}, x_1), (\mathsf{ff}, x_2)\}) \\
x_1 &= \mathsf{fold}(\{(\mathsf{tt}, x_0), (\mathsf{ff}, x_2)\}) \\
x_2 &= \mathsf{fold}(\{(\mathsf{tt}, x_0), (\mathsf{ff}, x_2)\})
\end{aligned}
$$



- How to ensure

$$(x = y) \simeq \mathsf{Bisim}(x, y)$$

Clocked Cubical Type Theory (CCTT)

- Represent $\mathsf{P_f}$ as higher-inductive type (HIT)
- Encode coinductive types using multiclocked guarded recursion

$$\mathsf{LTS}^{\kappa} \simeq \mathsf{P_f}(A \times \triangleright^{\kappa}\mathsf{LTS}^{\kappa}) \qquad \mathsf{LTS} \stackrel{\text{def}}{=} \forall\kappa.\mathsf{LTS}^{\kappa}$$

- Program and reason about coinductive data using guarded recursion
- Results in paper (Kristensen et al. [2022])
  - Definition of CCTT
  - Principle of induction under clocks
  - Computational contents to clock irrelevance
  - Denotational semantics
- Partially implemented in extension of Cubical Agda

Clocked Cubical Type Theory

Extend Cubical Type theory ...

- Path types

$$\frac{\Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \lambda i.\, t : \mathrm{Path}_A(t[0/i], t[1/i])}$$

$$\frac{\Gamma \vdash p : \mathrm{Path}_A(a_0, a_1) \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p\, r : A}$$

- Composition

$$\frac{\begin{array}{cc} \Gamma \vdash \varphi : \mathbb{F} & \Gamma, i : \mathbb{I} \vdash A \text{ type} \\ \Gamma, \varphi, i : \mathbb{I} \vdash u : A & \Gamma \vdash u_0 : A[0/i][\varphi \mapsto u[0/i]] \end{array}}{\Gamma \vdash \mathrm{comp}^i_A\, [\varphi \mapsto u]\, u_0 : A[1/i][\varphi \mapsto u[1/i]]}$$

- Glueing, HITs, ...

. . . with multiclocked guarded recursion

- Clocks

$$\frac{\Gamma, \kappa : \mathsf{clock} \vdash t : A}{\Gamma \vdash \lambda\kappa.t : \forall\kappa.A} \qquad \frac{\Gamma \vdash t : \forall\kappa.A \qquad \Gamma \vdash \kappa' : \mathsf{clock}}{\Gamma \vdash t[\kappa'] : A[\kappa'/\kappa]}$$

- Later type

$$\frac{\Gamma \vdash \kappa : \mathsf{clock} \qquad \alpha \notin \Gamma}{\Gamma, \alpha : \kappa \vdash} \qquad \frac{\Gamma, \alpha : \kappa \vdash t : A}{\Gamma \vdash \triangleright(\alpha : \kappa).A \ \mathsf{type}}$$

- Introduction and (simplified) elimination rules

$$\frac{\Gamma, \alpha : \kappa \vdash t : A}{\Gamma \vdash \lambda(\alpha : \kappa).t : \triangleright(\alpha : \kappa).A} \qquad \frac{\Gamma \vdash t : \triangleright(\alpha : \kappa).A}{\Gamma, \beta : \kappa, \Gamma' \vdash t\,[\beta] : A[\beta/\alpha]}$$

- Force

$$\forall\kappa. \triangleright^{\kappa} A \simeq \forall\kappa.A$$

Fixed points

$$\frac{\Gamma \vdash t : \triangleright^{\kappa} A \to A}{\Gamma \vdash \mathsf{dfix}^{\kappa}\, t : \triangleright^{\kappa} A}$$

$$\frac{\Gamma \vdash t : \triangleright^{\kappa} A \to A}{\Gamma \vdash \mathsf{pfix}^{\kappa}\, t : \triangleright(\alpha : \kappa).\mathsf{Path}_A((\mathsf{dfix}^{\kappa}\, t)\,[\alpha], t(\mathsf{dfix}^{\kappa}\, t))}$$

- **Lemma:** The type $\Sigma(x : A).\mathsf{Path}_A(x, f\,(\lambda(\alpha : \kappa).x))$ is contractible for every $f : \triangleright^{\kappa} A \to A$.
- Nakano fixed point operator

$$\mathsf{fix}^{\kappa} \stackrel{\text{def}}{=} \lambda f. f(\mathsf{dfix}^{\kappa} f) : (\triangleright^{\kappa} A \to A) \to A$$

Example: Streams

- Guarded stream type

$$\overline{\mathsf{Str}^{\kappa}}(\mathbb{N}) \stackrel{\text{def}}{=} \mathsf{fix}^{\kappa}(\lambda X.\overline{\mathbb{N}}\,\overline{\times}\,\overline{\triangleright}\,(\alpha:\kappa).X\,[\alpha]) \qquad : \mathsf{U}$$

$$\mathsf{Str}^{\kappa}(\mathbb{N}) \stackrel{\text{def}}{=} \mathsf{El}\,(\overline{\mathsf{Str}^{\kappa}}(\mathbb{N}))$$

- Using

$$\frac{\Gamma, \alpha : \kappa \vdash A : \mathsf{U}}{\Gamma \vdash \overline{\triangleright}\,(\alpha:\kappa).A : \mathsf{U}}$$

- Then

$$\mathsf{Str}^{\kappa}(\mathbb{N}) \simeq \mathbb{N} \times \triangleright^{\kappa}\mathsf{Str}^{\kappa}(\mathbb{N})$$

- Recursive programs

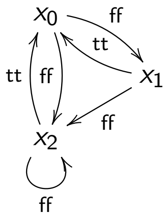
$$\mathsf{zeros} \stackrel{\text{def}}{=} \mathsf{fix}^{\kappa}(\lambda xs.0 :: xs)$$

Example: Guarded LTS

- Guarded LTS type

$$\mathsf{LTS}^\kappa \simeq \mathsf{P_f}(A \times \triangleright^\kappa \mathsf{LTS}^\kappa)$$

- Guarded recursive definitions



$$\begin{aligned}\mathsf{fix}^\kappa(\lambda x : \triangleright^\kappa(\mathsf{LTS}^\kappa)^3.(&\mathsf{fold}(\{(\mathsf{ff}, \triangleright^\kappa(\pi_1)(x)), (\mathsf{ff}, \triangleright^\kappa(\pi_2)(x))\}),\\ &\mathsf{fold}(\{(\mathsf{tt}, \triangleright^\kappa(\pi_0)(x)), (\mathsf{ff}, \triangleright^\kappa(\pi_2)(x))\}),\\ &\mathsf{fold}(\{(\mathsf{tt}, \triangleright^\kappa(\pi_0)(x), (\mathsf{ff}, \triangleright^\kappa(\pi_2)(x)))\})))\end{aligned}$$

Encoding coinductive types

Encoding coinductive types

- **Definition.** A functor[1] $F : (I \to U) \to (I \to U)$ *commutes with clock quantification* if

$$F(\forall \kappa. X) \simeq \forall \kappa. F(X)$$

- **Definition.** $g : Y \to F(Y)$ is a *final coalgebra* if the following type is contractible for all $f : X \to F(X)$

$$\Sigma(h : X \to Y).\, g \circ h = F(h) \circ f$$

- **Theorem.** If F commutes with clock quantification, then $\nu(F)$ is final coalgebra

$$\nu^\kappa(F) \simeq F(\triangleright^\kappa(\nu^\kappa(F))$$

$$\nu(F) \stackrel{\text{def}}{=} \forall \kappa. \nu^\kappa(F)$$

[1] in the naive sense

Encoding coinductive types

- **Theorem.** If F commutes with clock quantification, then $\nu(F)$ is final coalgebra

$$\nu^\kappa(F) \simeq F(\triangleright^\kappa(\nu^\kappa(F))$$

$$\nu(F) \stackrel{\text{def}}{=} \forall\kappa.\nu^\kappa(F)$$

- Note

$$\begin{aligned}\nu(F) &\simeq \forall\kappa.F(\triangleright^\kappa \nu^\kappa(F)) \\ &\simeq F(\forall\kappa.\triangleright^\kappa \nu^\kappa(F)) \\ &\simeq F(\forall\kappa.\nu^\kappa(F)) \\ &= F(\nu(F))\end{aligned}$$

Example: Streams

- Functor

$$F(X) = \mathbb{N} \times X$$

- Type equivalences

$$\begin{aligned} \forall\kappa.F(X) &= \forall\kappa.(\mathbb{N} \times X) \\ &\simeq (\forall\kappa.\mathbb{N}) \times (\forall\kappa.X) \end{aligned}$$

- Need $\mathbb{N}$ clock irrelevant

$$\mathbb{N} \simeq \forall\kappa.\mathbb{N}$$

Examples

- The delay monad $LA \simeq A + LA$
- Functor

$$F(X) = A + X$$

- Equivalence

$$\begin{aligned}\forall\kappa.F(X) &= \forall\kappa.(A + X)\\ &\simeq (\forall\kappa.A) + (\forall\kappa.X)\\ &\simeq A + \forall\kappa.X\\ &= F(\forall\kappa.X)\end{aligned}$$

- So need A clock irrelevant and

$$\forall\kappa.(X + Y) \simeq (\forall\kappa.X) + (\forall\kappa.Y)$$

Example: Non-deterministic processes

- Encoding non-deterministic processes

$$\mathsf{LTS}^\kappa \simeq \mathsf{P_f}(A \times \triangleright^\kappa \mathsf{LTS}^\kappa) \qquad \mathsf{LTS} \overset{\mathrm{def}}{=} \forall\kappa.\mathsf{LTS}^\kappa$$

- Requires

$$\begin{aligned}\forall\kappa.\mathsf{P_f}(A \times X) &\simeq \mathsf{P_f}(\forall\kappa.(A \times X)) \\ &\simeq \mathsf{P_f}((\forall\kappa.A) \times (\forall\kappa.X)) \\ &\simeq \mathsf{P_f}(A \times (\forall\kappa.X))\end{aligned}$$

Induction under clocks

Induction under clocks

- New principle for HITs
- Case of Bool:

$$\frac{\Gamma, x : \forall \vec{\kappa}.\mathsf{Bool} \vdash C(x)\ \mathsf{type} \qquad \Gamma \vdash u_{\mathsf{tt}} : C(\lambda \vec{\kappa}.\mathsf{tt}) \qquad \Gamma \vdash u_{\mathsf{ff}} : C(\lambda \vec{\kappa}.\mathsf{ff}) \qquad \Gamma \vdash t : \forall \vec{\kappa}.\mathsf{Bool}}{\Gamma \vdash \mathsf{elim}_C(u_{\mathsf{tt}}, u_{\mathsf{ff}}, t) : C(t)}$$

- Plus definitional equalities

$$\mathsf{elim}_C(u_{\mathsf{tt}}, u_{\mathsf{ff}}, \lambda \vec{\kappa}.\mathsf{tt}) \equiv u_{\mathsf{tt}}$$
$$\mathsf{elim}_C(u_{\mathsf{tt}}, u_{\mathsf{ff}}, \lambda \vec{\kappa}.\mathsf{ff}) \equiv u_{\mathsf{ff}}$$

Proving Bool clock irrelevant

$$\frac{\Gamma, x : \forall\vec{\kappa}.\mathrm{Bool} \vdash C(x)\ \mathrm{type} \qquad \Gamma \vdash u_{\mathrm{tt}} : C(\lambda\vec{\kappa}.\mathrm{tt}) \qquad \Gamma \vdash u_{\mathrm{ff}} : C(\lambda\vec{\kappa}.\mathrm{ff}) \qquad \Gamma \vdash t : \forall\vec{\kappa}.\mathrm{Bool}}{\Gamma \vdash \mathrm{elim}_C(u_{\mathrm{tt}}, u_{\mathrm{ff}}, t) : C(t)}$$

- Used to construct terms

$$\lambda x.\lambda\kappa.x : \mathrm{Bool} \to \forall\kappa.\mathrm{Bool}$$
$$\lambda x.\mathrm{elim}_{\mathrm{Bool}}(\mathrm{tt}, \mathrm{ff}, x) : (\forall\kappa.\mathrm{Bool}) \to \mathrm{Bool}$$

- and prove Bool clock irrelevant

$$\lambda x.\mathrm{elim}(\mathrm{refl}, \mathrm{refl}, x) : \Pi(x : \forall\kappa.\mathrm{Bool}).x = \lambda\kappa.\mathrm{elim}_{\mathrm{Bool}}(\mathrm{tt}, \mathrm{ff}, x)$$

Induction under clocks for $\mathbb{N}$

$$\frac{\Gamma, x : \forall\kappa.\mathbb{N} \vdash C(x) \text{ type} \qquad \Gamma \vdash u_0 : C(\lambda\kappa.0) \\ \Gamma, x : \forall\kappa.\mathbb{N}, y : C(x) \vdash u_s : C(\lambda\kappa.s(x\,[\kappa])) \qquad \Gamma \vdash t : \forall\kappa.\mathbb{N}}{\Gamma \vdash \mathsf{elim}_C(u_0, u_s, t) : C(t)}$$

- Plus definitional equalities

$$\begin{aligned} \mathsf{elim}_C(u_0, u_s, \lambda\kappa.0) &\equiv u_0 \\ \mathsf{elim}_C(u_0, u_s, \lambda\kappa.s(n)) &\equiv u_s(\lambda\kappa.n, \mathsf{elim}_C(u_0, u_s, \lambda\kappa.n)) \end{aligned}$$

- Used to prove $\mathbb{N}$ clock-irrelevant

Induction under clocks for spheres

$$\mathsf{base} : \mathbb{S}^n$$

$$\mathsf{surface} : (\bar{i} : \mathbb{I}^n) \to \mathbb{S}^n \left[\bigvee_{0 \le k < n} (i_k = 0 \vee i_k = 1) \mapsto \mathsf{base} \right]$$

- Induction under clocks

$$\frac{\Gamma, x : \forall\kappa.\mathbb{S}^n \vdash D \text{ type} \quad \Gamma \vdash u_b : D[\lambda\kappa.\mathsf{base}] \quad \Gamma \vdash t : \forall\kappa.\mathbb{S}^n \qquad \Gamma, \bar{i} : \mathbb{I}^n \vdash u_s : D[\lambda\kappa.\mathsf{surface}(\bar{i})] \left[\bigvee_{0 \le k < n} (i_k = 0 \vee i_k = 1) \mapsto u_b \right]}{\Gamma \vdash \mathsf{elim}_D(u_b, u_s, t) : D[t]}$$

- Definitional equalities

$$\mathsf{elim}_D(u_b, u_s, \lambda\kappa.\mathsf{base}) = u_b$$
$$\mathsf{elim}_D(u_b, u_s, \lambda\kappa.\mathsf{surface}(\bar{i})) = u_s$$

Spheres clock irrelevant

- Induction under clocks

$$\frac{\Gamma, x : \forall\kappa.\mathbb{S}^n \vdash D \text{ type} \quad \Gamma \vdash u_b : D[\lambda\kappa.\text{base}] \quad \Gamma \vdash t : \forall\kappa.\mathbb{S}^n \qquad \Gamma, \bar{i} : \mathbb{I}^n \vdash u_s : D[\lambda\kappa.\text{surface}(\bar{i})] \left[\bigvee_{0 \le k < n} (i_k = 0 \vee i_k = 1) \mapsto u_b \right]}{\Gamma \vdash \text{elim}_D(u_b, u_s, t) : D[t]}$$

- Equivalence

$$\lambda x.\lambda\kappa.x : \mathbb{S}^n \to \forall\kappa.\mathbb{S}^n$$

$$\lambda x.\text{elim}_D(\text{base}, \text{surface}(\bar{i}), x) : \forall\kappa.\mathbb{S}^n \to \mathbb{S}^n$$

Propositional truncation

$$\mathsf{in} : A \to \|A\|_{-1}$$

$$\mathsf{squash} : (a_0, a_1 : \|A\|_{-1}) \to (i : \mathbb{I}) \to \|A\|_{-1} \left[\begin{array}{l} i = 0 \mapsto a_0 \\ i = 1 \mapsto a_1 \end{array} \right]$$

- Induction under clocks

$$\frac{\begin{array}{c} \Gamma \vdash A : \forall\kappa.\mathsf{U} \quad \Gamma, x : \forall\kappa.\|A\,[\kappa]\|_{-1} \vdash D \text{ type} \\ \Gamma, x : \forall\kappa.A \vdash u_{\mathsf{in}} : D[\lambda\kappa.\mathsf{in}(x\,[\kappa])] \qquad \Gamma \vdash t : \forall\kappa.\|A\,[\kappa]\|_{-1} \\ \Gamma, a_0, a_1 : \forall\kappa.\|A\,[\kappa]\|_{-1}, y_0 : D[a_0], y_1 : D[a_1], i : \mathbb{I} \\ \vdash u_{\mathsf{sq}} : D[\lambda\kappa.\mathsf{squash}(a_0\,[\kappa], a_1\,[\kappa], i)] \left[\begin{array}{l} i = 0 \mapsto y_0 \\ i = 1 \mapsto y_1 \end{array} \right] \end{array}}{\Gamma \vdash \mathsf{elim}_D(u_{\mathsf{in}}, u_{\mathsf{squash}}, t) : D[t]}$$

Truncation commutes with clock abstraction

$$\frac{\begin{array}{c}\Gamma \vdash A : \forall\kappa.\mathsf{U} \quad \Gamma, x : \forall\kappa.\|A\,[\kappa]\|_{-1} \vdash \|\forall\kappa.A\,[\kappa]\|_{-1} \text{ type} \\ \Gamma, x : \forall\kappa.A \vdash \mathsf{in}(x) : \|\forall\kappa.A\,[\kappa]\|_{-1} \qquad \Gamma \vdash t : \forall\kappa.\|A\,[\kappa]\|_{-1} \\ \Gamma, a_0, a_1 : \forall\kappa.\|A\,[\kappa]\|_{-1}, y_0 : \|\forall\kappa.A\,[\kappa]\|_{-1}, y_1 : \|\forall\kappa.A\,[\kappa]\|_{-1}, i : \mathbb{I} \\ \vdash \mathsf{squash}(y_0, y_1, i) : \|\forall\kappa.A\,[\kappa]\|_{-1} \begin{bmatrix} i = 0 \mapsto y_0 \\ i = 1 \mapsto y_1 \end{bmatrix}\end{array}}{\Gamma \vdash \mathsf{elim}_D(\mathsf{in}, \mathsf{squash}, t) : \|\forall\kappa.A\,[\kappa]\|_{-1}}$$

- Equivalence

$$\lambda x.\lambda\kappa.\|(-)\,[\kappa]\|_{-1}(x) : \|\forall\kappa.A\,[\kappa]\|_{-1} \to \forall\kappa.\|A\,[\kappa]\|_{-1}$$
$$\mathsf{elim}_D(\mathsf{in}, \mathsf{squash}, -) : \forall\kappa.\|A\,[\kappa]\|_{-1} \to \|\forall\kappa.A\,[\kappa]\|_{-1}$$

- Generalises to higher truncations as well

Finite powersets as a HIT

$$
\begin{aligned}
\emptyset &: \mathsf{P_f}(A) \\
\{-\} &: A \to \mathsf{P_f}(A) \\
\cup &: \mathsf{P_f}(A) \to \mathsf{P_f}(A) \to \mathsf{P_f}(A) \\
\mathsf{nl} &: \Pi(X : \mathsf{P_f}(A)).\, X \cup \emptyset = X \\
\mathsf{assoc} &: \Pi(X\ Y\ Z : \mathsf{P_f}(A)).\, X \cup (Y \cup Z) = (X \cup Y) \cup Z \\
\mathsf{comm} &: \Pi(X\ Y : \mathsf{P_f}(A)).\, X \cup Y = Y \cup X \\
\mathsf{idem} &: \Pi(X : \mathsf{P_f}(A)).\, X \cup X = X \\
\mathsf{trunc} &: \mathsf{isSet}(\mathsf{P_f}(A))
\end{aligned}
$$

Induction under clocks for $\mathsf{P_f}$

- Assuming $\Gamma \vdash X : \forall\kappa.\mathsf{U}$

$$\frac{\begin{array}{c}\Gamma, x : \forall\kappa.\mathsf{P_f}(X\,[\kappa]) \vdash C(x)\ \mathsf{type} \qquad \Gamma \vdash u_\emptyset : C(\lambda\kappa.\emptyset) \\ \Gamma, x : \forall\kappa.(X\,[\kappa]) \vdash u_{\{-\}} : C(\lambda\kappa.\{x\,[\kappa]\}) \\ \Gamma, x, x' : \forall\kappa.\mathsf{P_f}(X\,[\kappa]), y : C(x), y' : C(x') \vdash u_\cup : C(\lambda\kappa.x\,[\kappa] \cup x'\,[\kappa]) \\ \dots \\ \Gamma, x : \forall\kappa.\mathsf{P_f}(A\,[\kappa]), y : C(x), i : \mathbb{I} \vdash u_{\mathsf{idem}} : C(\lambda\kappa.\mathsf{idem}(x\,[\kappa], i)) \\ u_{\mathsf{idem}}(x, y, 0) \equiv u_\cup(x, x, y, y) \qquad u_{\mathsf{idem}}(x, y, 1) \equiv y \\ \dots \qquad \Gamma \vdash t : \forall\kappa.\mathsf{P_f}(X\,[\kappa])\end{array}}{\Gamma \vdash \mathsf{elim}_C(u_\emptyset, u_{\{-\}}, \dots, t) : C(t)}$$

- Can be used to prove

$$\forall\kappa.\mathsf{P_f}(X\,[\kappa]) \simeq \mathsf{P_f}(\forall\kappa.X\,[\kappa])$$

Bisimilarity as guarded recursive type

- Let $x, y : \mathsf{LTS}$

$$\mathsf{Bisim}^{\kappa}(x, y) \stackrel{\text{def}}{=} \mathsf{Sim}^{\kappa}(x, y) \times \mathsf{Sim}^{\kappa}(y, x)$$

- Where

$$\begin{aligned}\mathsf{Sim}^{\kappa}(x, y) \simeq \Pi(x' : \mathsf{LTS}, a : A).(a, x') \in \mathsf{ufld}(x) \to \\ \exists y' : \mathsf{LTS}.(a, y') \in \mathsf{ufld}(y) \times \triangleright(\alpha : \kappa).\mathsf{Bisim}^{\kappa}(x', y')\end{aligned}$$

- Then coinductive bisimilarity for $x, y : \mathsf{LTS}$ is

$$\mathsf{Bisim}(x, y) \stackrel{\text{def}}{=} \forall \kappa.\mathsf{Bisim}^{\kappa}(x, y)$$

- **Theorem.** $(x = y) \simeq \mathsf{Bisim}(x, y)$

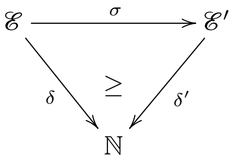
Denotational semantics

Denotational semantics

- Cubical Type Theory modelled in $\mathrm{PSh}(\mathscr{C})$
- Clocked Type Theory modelled in $\mathrm{PSh}(\mathscr{T})$
- Category $\mathscr{T}$ of time objects
  - Objects:

$$(\mathscr{E}, \delta : \mathscr{E} \to \mathbb{N})$$

  - $\mathscr{E}$ finite set (of clocks)
  - Morphisms $\sigma : (\mathscr{E}, \delta) \to (\mathscr{E}', \delta')$:



- Clocked Cubical Type Theory modelled in $\mathrm{PSh}(\mathscr{C} \times \mathscr{T})$

Model construction

- Model constructed in internal language following Licata et al. [2018]
- Interval type and cofibrations imported from $\mathrm{PSh}(\mathscr{C})$
- $\triangleright$ and ticks modelled using dependent right adjoint
- Composition structure on $\triangleright$ from general theorem proved in paper

Modelling guarded recursion

- Object of clocks

$$\mathrm{Clk}(I, \mathscr{E}, \delta) \stackrel{\mathsf{def}}{=} \mathscr{E}$$

- $\triangleright$ is a modal operator on slice

$$\mathsf{PSh}(\mathscr{C} \times \mathscr{T})/\mathrm{Clk}$$

- Defined on closed types as

$$\triangleright A(I, \mathscr{E}, \delta, \lambda) = \begin{cases} A(I, \mathscr{E}, \delta[\lambda \mapsto n], \lambda) & \text{if } \delta(\lambda) = n+1 \\ 1 & \text{if } \delta(\lambda) = 0 \end{cases}$$

- Fixed points defined by natural number induction

Quantification over clocks

- Object of clocks
$$\mathrm{Clk}(I, \mathscr{E}, \delta) \stackrel{\mathsf{def}}{=} \mathscr{E}$$
- $(\Pi(\kappa : \mathrm{Clk}).A)(I, \mathscr{E}, \delta)$ isomorphic to limit of sequence
$$A(I, (\mathscr{E}, \lambda), \delta[\lambda \mapsto 0]) \leftarrow A(I, (\mathscr{E}, \lambda), \delta[\lambda \mapsto 1]) \leftarrow \dots$$
- So
$$\Pi(\kappa : \mathrm{Clk}).(A + B) \simeq (\Pi(\kappa : \mathrm{Clk}).A) + (\Pi(\kappa : \mathrm{Clk}).B)$$

Modelling induction under clocks

- Model HITs following Coquand et al. [2018]
- This means each element in $H(I, \mathscr{E}, \delta)$ is either
  - A constructor for H (including those for paths), or
  - An hcomp
- Each map in diagram

$$H(I, (\mathscr{E}, \lambda), \delta[\lambda \mapsto 0]) \leftarrow H(I, (\mathscr{E}, \lambda), \delta[\lambda \mapsto 1]) \leftarrow \dots$$

- preserves this structure strictly
- This allows us to prove soundness of induction under clocks principle

Coinductive types in model

- To commute with $\forall\kappa$ means to commute with ω-limits
- Coinductive types interpreted as limits

$$F(1)(I,\mathscr{E},\delta) \leftarrow F^2(1)(I,\mathscr{E},\delta) \leftarrow F^3(1)(I,\mathscr{E},\delta) \leftarrow \dots$$

- Also case of $F = \mathsf{P_f}$ is ω-limit
- In set theory final coalgebra for $\mathsf{P_f}$ is constructed in $\omega + \omega$ steps (Worrell [2005])
- Veltri [2021] internalises Worrell's construction in Cubical Agda

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- Also case of $F = \mathrm{P_f}$ is ω-limit
- In set theory final coalgebra for $\mathrm{P_f}$ is constructed in $\omega + \omega$ steps (Worrell [2005])
- Veltri [2021] internalises Worrell's construction in Cubical Agda
- $\mathrm{PSh}(\mathscr{C} \times \mathscr{T})$ is a category of **strict** cubical presheaves over $\mathscr{T}$
- Cobar construction would not model equivalence

$$\mathrm{P_f}(\forall\kappa.-) \simeq \forall\kappa.\mathrm{P_f}(-)$$

Beyond coinduction

Models of untyped lambda calculus with non-determinism

- Two guarded powerdomains

$$\mathsf{P}^{\kappa}_{\Diamond}(A) \simeq \mathsf{P}_{\mathsf{f}}(A + \triangleright^{\kappa}\mathsf{P}^{\kappa}_{\Diamond}(A))$$
$$\mathsf{P}^{\kappa}_{\Box}(A) \stackrel{\text{def}}{=} \mathsf{L}^{\kappa}\mathsf{P}_{\mathsf{f}}(A)$$

- Denotational model ($T = \mathsf{P}^{\kappa}_{\Diamond}$ or $\mathsf{P}^{\kappa}_{\Box}$)

$$\mathsf{SVal}^{\kappa} \stackrel{\text{def}}{=} (\triangleright^{\kappa}(\mathsf{SVal}^{\kappa} \to T(\mathsf{SVal}^{\kappa})))$$

- Used to prove applicative bisimilarity a congruence
- Details in (Møgelberg and Vezzosi [2021])
- Other applications: Models of higher-order store (Sterling et al. [2022])

Conclusion

Results

- General theorem for encoding of coinductive types
- Induction under clocks formulated for general schema (Cavallo and Harper [2019]) for HITs
- Denotational semantics in $\mathsf{PSh}(\mathscr{C} \times \mathscr{T})$
- Give computational content to *tick irrelevance axiom*

$$\mathsf{tirr}^{\kappa} : \Pi(x : \triangleright^{\kappa} A).\triangleright(\alpha : \kappa).\triangleright(\beta : \kappa).\mathsf{Path}_A(x\,[\alpha], x\,[\beta])$$

- **Conjecture (Canonicity).** Any term $\vec{i} : \mathbb{I}, \vec{\kappa} : \mathsf{clock} \vdash t : \mathbb{N}$ is equal to either 0 or a successor.

Future work

- Clock-irrelevant universes of clock-irrelevant types
- Proving canonicity
- Implementation of induction under clocks in Agda
- Start using Guarded Cubical Agda for coinduction and program verification!

References I

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