

Every Modality is a Relative Right Adjoint

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EuroProofNet WG6 Meeting

Vienna, Austria

April 24, 2023

Let $R : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

$$\frac{\Gamma \text{ ctx } @ \mathcal{C}}{R\Gamma \text{ ctx } @ \mathcal{D}} \quad \frac{\tau : \Gamma \rightarrow \Gamma' @ \mathcal{C}}{R\tau : R\Gamma \rightarrow R\Gamma' @ \mathcal{D}} \quad \frac{\Gamma \vdash T \text{ type } @ \mathcal{C}}{R\Gamma \vdash RT \text{ type } @ \mathcal{D}} \quad \frac{\Gamma \vdash t : T @ \mathcal{C}}{R\Gamma \vdash Rt : RT @ \mathcal{D}}$$

Ok, so how do we check

$$\frac{?}{\Delta \vdash RT \text{ type}}$$

We check $\Gamma \vdash T \text{ type } @ \mathcal{C}$ and substitute with $\sigma : \Delta \rightarrow R\Gamma$.

BUT: Don't bother the user. Synthesize Γ and σ .

$\Gamma \in \mathcal{C}$ should be the **universal** context Γ such that $\sigma : \Delta \rightarrow R\Gamma$ exists.

I.e. if $\sigma' : \Delta \rightarrow R\Gamma'$ then we should have $\Gamma \rightarrow \Gamma'$.

+ some sensible laws $\rightsquigarrow L \dashv R$.

Let $R : \mathcal{C} \rightarrow \mathcal{D}$ be a CwF morphism.

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MTT [GKNB21] is parametrized by a **2-category**:

- modes p, q, r, \dots
- modalities $\mu : p \rightarrow q$

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$$\frac{\Gamma, \mu \vdash T \text{type} @ p}{\Gamma \vdash \langle \mu \mid T \rangle \text{type} @ q}$$

$$\frac{\Gamma, \mu \vdash t : T @ p}{\Gamma \vdash \text{mod}_{\mu} t : \langle \mu \mid T \rangle @ q}$$

- (2-cells $\alpha : \mu \Rightarrow \nu$).

Semantics:

- $\llbracket p \rrbracket$ is a (often presheaf) category modelling all of DTT,
- $\llbracket \mu \rrbracket$ is a (weak) dependent right adjoint (DRA) [BCMMPS20] to $\llbracket \mu \rrbracket$,

Note: If codomain \mathcal{D} is democratic, then DRA = right adjoint that is a CwF morphism.

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*“A more serious and mathematical issue is that **MTT requires all modalities to be right adjoints**, semantically, because you have to have some operation to interpret the locking functors on contexts. (And FitchTT even requires those left adjoints to themselves be (parametric) right adjoints.) This seems a **serious restriction on the kinds of situations we can model.**”*

— Mike Shulman, HoTT mailing list, Dec 1, 2022 (emphases are ours)

- Valid concern: We can **internally** prove that MTT modalities preserve limits, e.g. $\langle \mu \mid A \times B \rangle \cong \langle \mu \mid A \rangle \times \langle \mu \mid B \rangle$.
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Multimodal Adjoint Type Theory (MATT):

[Shulman, March 2023]

Categorically Adds locks to contexts
cofreely.

Morally ~~Defines locks by induction on~~
syntactic context formation.
These **approximate** the left
adjoint.

☺ $\langle \mu \mid - \rangle$ need not be a DRA.

☺ Subsumes MTT without
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This is also assumed in the
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[Shu23, assumption 4.1]

Our solution (WIP):

Categorically Move to **copresheaf** category.

Morally Move to **metaprogramming
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Presheaves:

$$\mathbf{Psh}(\mathcal{C}) = [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

Swap & curry Hom : $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$
to get $\mathbf{y} : \mathcal{C} \rightarrow \mathbf{Psh}(\mathcal{C}) : \Gamma \mapsto \text{Hom}(-, \Gamma)$

Functor $F : \mathcal{C} \rightarrow \mathcal{D}$ yields
 $F_! \dashv F^* \dashv F_* : \mathbf{Psh}(\mathcal{C}) \rightarrow \mathbf{Psh}(\mathcal{D})$
where $F_!$ extends F :

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Every functor is a left/right-relative left/right adjoint

Presheaves:

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This is a left-relative adjunction:

$$F_{\mathbf{y}} \dashv F^* \mathbf{y}$$

Copresheaves:

$$\begin{aligned}
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This is a right-relative adjunction:

$$F^\circ \mathbf{h} \dashv_{\mathbf{h}} F$$

$$\frac{\mathbf{h}\Gamma, \bar{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \text{Copsh}(\mathcal{C})}{\Gamma \vdash \text{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathcal{D}}$$

where $\llbracket \bar{\mu} \rrbracket = \llbracket \mu \rrbracket^\circ$

Every functor is a left/right-relative left/right adjoint

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As of this point,
things are going downhill.

Thoughts & ideas appreciated.

So surely, h is well-behaved?

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In non-pathological situations:

- h is **never** a DRA,
- h **never** preserves limits,
 $\langle \mathbf{h} \mid A \times B \rangle \not\rightarrow \langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid B \rangle$
- h is **never** applicative,
 $\langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid A \rightarrow C \rangle \not\rightarrow \langle \mathbf{h} \mid A \times (A \rightarrow C) \rangle$

$\leadsto h$ is an MTT-unsupportive sediment.

To use a variable:

$$\frac{\mathbf{h}(\Gamma, \mathbf{v} \mid x : T), \bar{\mu}_\mu \vdash ? : \langle \mathbf{h} \mid T \rangle}{\Gamma, \mathbf{v} \mid x : T \vdash \text{mod}_\mu^{\mathbf{h}} ? : \langle \mu \mid T \rangle}$$

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So surely, $\text{Copsh}(\mathcal{C})$ is well-behaved?

$\text{Copsh}(\mathcal{C})$ is a CwF.

Giraud CwF structure [Gir65, BCMMP20]

Every category \mathcal{D} with \top and pullbacks is a CwF:

- Contexts and substitutions: \mathcal{D}
- $T \in \text{Ty}(\Gamma)$:



- Substitution
- Context extension

However, $\text{Copsh}(\mathcal{C})$ has:

- 🚫 **No Π -types!**
So no library functions!
- 😐 (We have co-exponentials.)
 $(A_E \rightarrow B) \cong (A \rightarrow E \uplus B)$
- 😞 **No universe?**

Possible solution:

Move to $\text{Psh}(\text{Copsh}(\mathcal{C}))$.

(Is this getting out of hand?)

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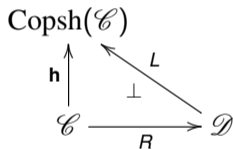
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We do not always need copresheaves.

It doesn't have to be a relative right adjoint along h .

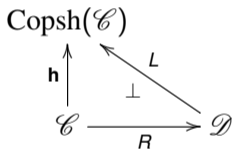


$$\text{Hom}_{\text{Copsh}(\mathcal{C})}(Ld, \mathbf{h}c) \cong \text{Hom}_{\mathcal{D}}(d, Rc)$$

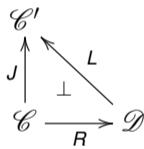


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We do not always need copresheaves.
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$$\text{Functor } F : \mathcal{C} \rightarrow \mathcal{D}$$

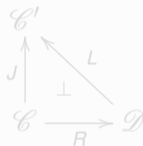
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Right multi-adjoint

PRA without referring to \top .

Relative right adjoint



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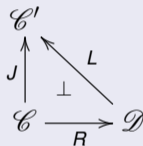
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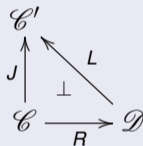
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Inspired by, but different from [GCKGB22].

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- **Does anyone need this generality?**

Thanks!

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Questions?

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