Every Modality is a Relative Right Adjoint

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Let $R : \mathscr{C} \to \mathscr{D}$ be a functor.

$$\frac{\Gamma \operatorname{ctx} @ \mathscr{C}}{R\Gamma \operatorname{ctx} @ \mathscr{D}} \qquad \frac{\tau : \Gamma \to \Gamma' @ \mathscr{C}}{R\tau : R\Gamma \to R\Gamma' @ \mathscr{D}} \qquad \frac{\Gamma \vdash T \operatorname{type} @ \mathscr{C}}{R\Gamma \vdash RT \operatorname{type} @ \mathscr{D}} \qquad \frac{\Gamma \vdash t : T @ \mathscr{C}}{R\Gamma \vdash Rt : RT @ \mathscr{D}}$$

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We check $\Gamma \vdash T \operatorname{type} @ \mathscr{C}$ and substitute with $\sigma : \Delta \to R\Gamma$.
BUT: Don't bother the user. Synthesize Γ and σ.

$\Gamma \in \mathscr{C}$ should be the **universal** context Γ such that $\sigma : \Delta \to R\Gamma$ exists.
I.e. if $\sigma' : \Delta \to R\Gamma'$ then we should have $\Gamma \to \Gamma'$.

+ some sensible laws $\rightsquigarrow L \dashv R$.

Let $R : \mathscr{C} \to \mathscr{D}$ be a CwF morphism.

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- modes $p, q, r, \ldots$
- modalities $\mu : p \to q$

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- (2-cells $\alpha : \mu \Rightarrow \nu$).

Semantics:

- $[\![p]\!]$ is a (often presheaf) category modelling all of DTT,
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Note: If codomain $\mathscr{D}$ is democratic, then DRA = right adjoint that is a CwF morphism.

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- Valid concern: We can **internally** prove that MTT modalities preserve limits, e.g. $\langle \mu \mid A \times B \rangle \cong \langle \mu \mid A \rangle \times \langle \mu \mid B \rangle$.
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Multimodal Adjoint Type Theory (MATT):
[Shulman, March 2023]

Categorically Adds locks to contexts **cofreely**.

Morally ~~Defines locks by induction on **syntactic context formation**.~~ These **approximate** the left adjoint.

☺ $\langle \mu \mid - \rangle$ need not be a DRA.

☺ Subsumes MTT without modifications.

☹ $\Rightarrow$ We can still **internally** prove that $\langle \mu \mid - \rangle$ preserves limits. This is also assumed in the **model**.
[Shu23, assumption 4.1]

Our solution (WIP):

Categorically Move to **copresheaf** category.

Morally Move to **metaprogramming with continuations**.

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☹ In particular, $\langle \mu \mid - \rangle$ may not:

- be a DRA,
- preserve limits,
- or even be applicative.

Great/Terrible!

Presheaves:

$\mathrm{Psh}(\mathscr{C}) = [\mathscr{C}^{\mathsf{op}}, \mathsf{Set}]$

Swap & curry $\mathrm{Hom} : \mathscr{C}^{\mathsf{op}} \times \mathscr{C} \to \mathsf{Set}$
to get $\mathbf{y} : \mathscr{C} \to \mathrm{Psh}(\mathscr{C}) : \Gamma \mapsto \mathrm{Hom}(-, \Gamma)$

Functor $F : \mathscr{C} \to \mathscr{D}$ yields
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Every functor is a left/right-relative left/right adjoint

Presheaves:

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& \mathrm{Hom}_{\mathscr{D}}(F\Delta, \Gamma) \\
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This is a left-relative adjunction:
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This is a right-relative adjunction:
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Every functor is a left/right-relative left/right adjoint

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As of this point,
things are going downhill.

Thoughts & ideas appreciated.

So surely, $\mathbf{h}$ is well-behaved?

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In non-pathological situations:

- $\mathbf{h}$ is **never** a DRA,
- $\mathbf{h}$ **never** preserves limits,

  $$\langle \mathbf{h} \mid A \times B \rangle \xrightarrow{\ncong} \langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid B \rangle$$

- $\mathbf{h}$ is **never** applicative.

  $$\langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid A \to C \rangle \nrightarrow \langle \mathbf{h} \mid A \times (A \to C) \rangle$$

$\rightsquigarrow$ $\mathbf{h}$ is an MTT-unsupportive sediment.

To use a variable:

$$\frac{\mathbf{h}(\Gamma, \nu \mid x : T), \overline{\text{🔒}}_{\mu} \vdash ? : \langle \mathbf{h} \mid T \rangle}{\Gamma, \nu \mid x : T \vdash \mathrm{mod}^{\mathbf{h}}_{\mu}\, ? : \langle \mu \mid T \rangle}$$

we need

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So surely, $\mathrm{Copsh}(\mathscr{C})$ is well-behaved?

$\mathrm{Copsh}(\mathscr{C})$ is a CwF.

Giraud CwF structure [Gir65, BCMMPS20]

Every category $\mathscr{D}$ with $\top$ and pullbacks is a CwF:

- Contexts and substitutions: $\mathscr{D}$
- $T \in \mathrm{Ty}(\Gamma)$:

$$\begin{tikzcd} & \Gamma.T & \Delta' \arrow[d] \\ \Theta & \Gamma \arrow[r] & \Delta \end{tikzcd}$$

- Substitution
- Context extension

However, $\mathrm{Copsh}(\mathscr{C})$ has:

- No Π-types!
  So no library functions!
- (We have co-exponentials.)
  $(A_E \to B) \cong (A \to E \uplus B)$
- No universe?

Possible solution:
Move to $\mathrm{Psh}(\mathrm{Copsh}(\mathscr{C}))$.
(Is this getting out of hand?)

☺ This is 2LTT for $\mathrm{Copsh}(\mathscr{C})$. [ACKS17/23]

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So surely, $\mathrm{Copsh}(\mathscr{C})$ is well-behaved?

$\mathrm{Copsh}(\mathscr{C})$ is a CwF.

Giraud CwF structure [Gir65, BCMMPS20]

Every category $\mathscr{D}$ with $\top$ and pullbacks is a CwF:

- Contexts and substitutions: $\mathscr{D}$
- $T \in \mathrm{Ty}(\Gamma)$:

$$\begin{tikzcd} & \Gamma . T \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta' \arrow[d] \\ \Theta \arrow[r] & \Gamma \arrow[r] & \Delta \end{tikzcd}$$

- Substitution
- Context extension

However, $\mathrm{Copsh}(\mathscr{C})$ has:

- No Π-types!
  So no library functions!
- (We have co-exponentials.)
  $(A_E \to B) \cong (A \to E \uplus B)$
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Possible solution:
Move to $\mathrm{Psh}(\mathrm{Copsh}(\mathscr{C}))$.
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It doesn't have to be a relative right adjoint along **h**.

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Right multi-adjoint

PRA without referring to $\top$.

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