Internal Algebras via Sketches (Category Theory of Universes)

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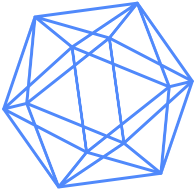

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What is Algebraic Topology?

Algebraic Topology involves a two step process:

1. Translating topological data (spaces, pointed spaces, spectra, sheaves of spaces, ...) to algebraic data (groups, modules, rings, ...)
2. Computing them

Standard examples includes homotopy groups, homology, cohomology, ...

Homological Algebra

Effective algebraic computations rely on a variety of advanced algebraic tools:

- Constructing Abelian Categories.
- Developing homological algebra.
- Using the resulting tools to compute cohomologies via exact sequences, spectral sequences, ...

Synthetic Algebraic Topology needs synthetic Algebra

Conclusion

Developing synthetic algebraic topology necessitates synthetic homological algebra!

Already pursued in type theory (van Doorn, Brunerie, ...):

1. Cohomology
2. Abelian category theory

Less can be found on the ∞-category side.

Homological Algebra in an ∞-Topos

This discussion motivates the following conjecture:

Conjecture

Let $\mathcal{E}$ be an ∞-topos. Then $\mathcal{A}\mathrm{b}(\mathcal{E})$, the 1-*category of abelian group objects* in $\mathcal{E}$, is an abelian category.

- What does it mean?
- Can we prove it?
- What kind of theory do we need?

Abelian Categories à la Grothendieck

- (AB 0) The category is additive, meaning enriched over abelian groups with finite biproducts.
- (AB 1) The category has (co)kernels
- (AB 2) The kernel of the cokernel coincides with the cokernel of the kernel.
- (AB 3) The category has all coproducts.
- (AB 3)* The category has all products.
- (AB 4) Coproducts are exact
- (AB 4)* Products are exact
- (AB 5) Filtered colimits are exact
- (AB 5)* Cofiltered limits are exact
- (AB 6) ...
- (AB 6)* ...

The presentable Case

Proposition

If $\mathcal{E}$ is a Grothendieck ∞-topos i.e. satisfies a presentability condition, then $\mathcal{A}\mathrm{b}(\mathcal{E})$ is abelian, satisfying (AB 1), (AB 2), (AB 3), (AB 4), (AB 5) and (AB 3).*

Proof.

A Grothendieck ∞-topos is given as a category of sheaves on some site, and so abelian group objects are given by sheaves of abelian groups, which is an abelian category with those properties. □

If $\mathcal{E}$ is not presentable, then everything beyond (AB 2) cannot even be articulated, as we do not have infinite colimits or limits.

Internalization of abelian Groups

The solution is to internalize things, which requires the following:

Conjecture

Let $\mathcal{E}$ be an ∞-topos. There exists a *universe of abelian groups* $\mathcal{U}_{\mathcal{A}\mathrm{b}}$, with the universal property $\mathrm{Map}(X, \mathcal{U}_{\mathcal{A}\mathrm{b}}) \simeq \mathcal{A}\mathrm{b}(\mathcal{E}_{/X})^{\simeq}$.

Has already been studied in various forms:

- Flaten[1]
- Hartwig[2]

[1] Jarl G. Taxerås Flaten. Univalent categories of modules. arXiv preprint, 2022. arXiv:2207.03261.

[2] Roswitha Harting. Internal coproduct of abelian groups in an elementary topos. Comm. Algebra, 10(11):1173–1237, 1982.

Internalization of Algebras

- The claim is not prohibitively difficult.
- Provides motivation for a general algebraic and diagrammatic internalization procedure.

Internalizing more generally

Conjecture

Let $\mathcal{E}$ be an ∞-topos and $\mathcal{A}$ an algebraic structure. There exists a *universe of algebraic objects* $\mathcal{U}_{\mathcal{A}}$, with the universal property $\mathrm{Map}(X, \mathcal{U}_{\mathcal{A}}) \simeq \mathcal{A}(\mathcal{E}_{/X})^{\simeq}$.

What does that mean?
How can we do that?

What is an algebraic Object?

Very broadly speaking an algebraic object is a collection of commutative diagrams, along with various product conditions. For example a monoid M is given as:

- An object M
- An element $e : 1 \to M$
- Multiplication $c : M \times M \to M$
- Associativity $c(c \times \mathrm{id}) = c(\mathrm{id} \times c)$
- Unitality $c(e \times \mathrm{id}) = c(\mathrm{id} \times e) = \mathrm{id}$

Objective

First classify general diagrams, and then impose various limit conditions.

What is the simplest non-trivial diagram?

Morphism Classifiers

We start with $\bullet \to \bullet$.

Lemma

Let $\mathcal{E}$ be an ∞-topos with object classifier $\mathcal{U}_{\mathcal{O}\mathrm{bj}}$. There exists a morphism classifier $\mathcal{U}_{\mathcal{M}\mathrm{or}}$, with the universal property $\mathrm{Map}_{\mathcal{E}}(X, \mathcal{U}_{\mathcal{M}\mathrm{or}}) \simeq (((\mathcal{E}_{/X})^{small})^{\Delta[1]})^{\simeq}$.

We can describe the morphism classifier explicitly:[3]

$$\mathcal{U}_{\mathcal{M}\mathrm{or}} = [\mathcal{U}_* \times \mathcal{U}_{\mathcal{O}\mathrm{bj}}, \mathcal{U}_{\mathcal{O}\mathrm{bj}} \times \mathcal{U}_*]_{\mathcal{U}_{\mathcal{O}\mathrm{bj}} \times \mathcal{U}_{\mathcal{O}\mathrm{bj}}}$$

using the Cartesian closure of $\mathcal{E}_{/\mathcal{U}_{\mathcal{O}\mathrm{bj}} \times \mathcal{U}_{\mathcal{O}\mathrm{bj}}}$ and the universal morphism $\mathcal{U}_* \to \mathcal{U}_{\mathcal{O}\mathrm{bj}}$.

[3] This is somewhat technical.

Source and Target

$\mathcal{U}_{\mathcal{M}\mathrm{or}}$ is by definition an object in $\mathcal{E}_{/\mathcal{U}_{\mathcal{O}\mathrm{bj}} \times \mathcal{U}_{\mathcal{O}\mathrm{bj}}}$, meaning it comes with maps $\mathcal{U}_{\mathcal{M}\mathrm{or}} \to \mathcal{U}_{\mathcal{O}\mathrm{bj}} \times \mathcal{U}_{\mathcal{O}\mathrm{bj}}$.

Remark

For every object X, this map induces the source-target map

$$(\mathcal{E}_{/X})^{\Delta[1]} \to \mathcal{E}_{/X} \times \mathcal{E}_{/X}$$

We hence use the notation

$$(s,t) : \mathcal{U}_{\mathcal{M}\mathrm{or}} \to \mathcal{U}_{\mathcal{O}\mathrm{bj}} \times \mathcal{U}_{\mathcal{O}\mathrm{bj}}$$

Composition and and Square Classifier

We can build on morphism classifiers to get further classifiers:

Composition classifier: $\mathcal{U}_{\mathcal{C}\mathrm{omp}} \simeq \mathcal{U}_{\mathcal{M}\mathrm{or}} \times_{\mathcal{U}_{\mathcal{O}\mathrm{bj}}} \mathcal{U}_{\mathcal{M}\mathrm{or}}$

Square classifier:

$$\begin{array}{ccc} \mathcal{U}_{\mathcal{S}\mathrm{q}} & \longrightarrow & \mathcal{U}_{\mathcal{C}\mathrm{omp}} \\ \downarrow & & \downarrow c \\ \mathcal{U}_{\mathcal{C}\mathrm{omp}} & \underset{c}{\longrightarrow} & \mathcal{U}_{\mathcal{M}\mathrm{or}} \end{array}$$

This approach is too ad-hoc. We want something more formal!

Internal ∞-Categories

Theorem (R.)

Let $\mathcal{C}$ be as above. Then the graph $\mathcal{U}_{\mathcal{M}\mathrm{or}} \rightrightarrows \mathcal{U}_{\mathcal{O}\mathrm{bj}}$ lifts to the structure of an internal ∞-category in $\mathcal{C}$, meaning a simplicial object $\underline{\mathcal{U}} : \Delta^{op} \to \mathcal{C}$ with

- $[0] \mapsto \mathcal{U}_{\mathcal{O}\mathrm{bj}}$
- $[1] \mapsto \mathcal{U}_{\mathcal{M}\mathrm{or}}$
- $[2] \mapsto \mathcal{U}_{\mathcal{C}\mathrm{omp}}$
- ...

General Diagram Classifier via left Kan extensions I

We now want to lift those classifying objects in a way that takes colimits to limits (as we saw with the square example). This is done via left Kan extension to the cocompletion under finite colimits:

$$\begin{array}{ccc} \mathbb{\Delta} & \xrightarrow{\underline{\mathcal{U}}^{op}} & \mathcal{C}^{op} \\ \downarrow & \nearrow_{\mathcal{U}_{(-)}} & \\ \widehat{\mathbb{\Delta}}_{\mathcal{F}\mathrm{in}} & & \\ \big\downarrow & & \\ \mathrm{Fun}(\mathbb{\Delta}^{op}, \mathcal{S}) & & \end{array}$$

General Diagram Classifier via left Kan extensions II

Theorem (R.–Rose)

Let I be a finite simplicial space. Then the object $\mathcal{U}_I$ is the internal I-shaped diagram classifier, meaning there is a natural equivalence

$$\mathrm{Map}(X, \mathcal{U}_I) \simeq (((\mathcal{E}_{/X})^I)^{small})^{\simeq}$$

Example

For example, taking $I = \Delta[1] \times \Delta[1]$, then $\mathcal{U}_I = \mathcal{U}_{\mathcal{S}\mathrm{q}}$ precisely classifies commutative squares.

Remark

The construction is nicely functorial i.e. $I \to J$ gives us restriction $\mathcal{U}_J \to \mathcal{U}_I$, which "restricts internal diagrams".

Diagrams are not good enough

We can now classify diagrams very efficiently, but recall that algebraic structures also have various limit conditions. So, we need to be able to add further limit conditions.

Limit Sketches

Definition

A limit sketch is a tuple $L = (D, \{D_i\}_{i \in I}, \{F_i\}_{i \in I})$ with D a simplicial space and for all $i \in I$, D_i a finite simplicial space and $F_i : D_i^{\triangleleft} \to D$. For a given finitely complete ∞-category $\mathcal{C}$, a model for a limit sketch is a functor $D \to \mathcal{C}$, such that for all $i \in I$ the restriction of the functor to $D_i^{\triangleleft}$ is a finite limit diagram in $\mathcal{C}$.

Remark

This differs somewhat from usual definitions, because we are not assuming our domain is a category.

Internalizing Limit Sketches

Theorem (R.–Rose)

Let $L = (D, \{D_i\}_{i\in I}, \{F_i\}_{i\in I})$ be a limit sketch, then there exists a subobject $\mathcal{U}_L$ of $\mathcal{U}_D$ classifying models of L, meaning there is a natural equivalence fitting into the following diagram

$$\begin{array}{ccc} \mathrm{Map}(X, \mathcal{U}_L) & \xrightarrow{\simeq} & \mathcal{M}\mathrm{od}_L(\mathcal{E}_{/X}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Map}(X, \mathcal{U}_D) & \xrightarrow{\simeq} & (((\mathcal{E}_{/X})^D)^{small})^{\simeq} \end{array}$$

Idea of Proof.

The internal ∞-category $\underline{\mathcal{U}}$ has finite limits which coincide with the larger category $\mathcal{E}$. Hence, we can obtain $\mathcal{U}_L$ via appropriately chosen restriction (i.e. pullbacks). □

Example I: Pointed Object Classifier

Example

Recall that $\mathcal{U}_{\mathcal{M}\mathrm{or}}$ classifies morphisms. How can we use it to classify pointed objects? Well, we simply take the following pullback

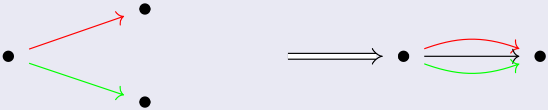
$$\begin{array}{ccc} \mathcal{U}_{\mathcal{P}\mathrm{t}} & \longrightarrow & \mathcal{U}_{\mathcal{M}\mathrm{or}} \\ \downarrow & & \downarrow s \\ 1 & \underset{*}{\longrightarrow} & \mathcal{U}_{\mathcal{O}\mathrm{bj}} \end{array}$$

to get the "pointed object classifier".

Example II: Multiplication Classifier

Example

Let $D = \{1,2,3\} \times \Delta^1/(\partial\Delta^1)$ (three parallel arrows with identified source and target). Let $\{1,3\}$ be the two point discrete category with functor $F : \{1,3\}^{\triangleleft} \to D$ given by sending the legs to the first and third arrow in D. We can depict this as follows:



Example II: Multiplication Classifier (Continued)

Example

A model of the limit sketch $L = (D, \{1, 3\}, F)$ is a multiplication map $M \times M \to M$ and its classifier is given by

$$\begin{array}{ccc} \mathcal{U}_{\mathcal{M}\mathrm{ult}} & \longrightarrow & \mathcal{U}_D \\ \downarrow & & \downarrow \mathcal{U}_F \\ \mathcal{U} & \xrightarrow[M \leftarrow M \times M \to M]{} & \mathcal{U}_\wedge \end{array},$$

meaning we have a "multiplication map classifier".

What is there left to do?

Several issues remain!

- The diagram classifier construction is by definition functorial, whereas the limit sketch classifiers are not functorial as described.

Main Challenge

Defining a good category of limit sketches!

For example if you demand that a functor

$$(D, \{D_i\}_{i \in I}, \{F_i\}_{i \in I}) \to (E, \{E_j\}_{j \in J}, \{G_j\}_{j \in J})$$

is given by a functor $D \to E$ such that for all i, $D_i^{\triangleleft} \to D \to E$ is part of the limit sketch, then the category does not have a nice terminal object.

What is there left to do?

- The description above is well-suited to deal with "externally finitary structures" like monoids, groups, ... We would like the possibility to understand "internally finite algebraic structures" that might not be externally well-behaved.

Main Challenge

Making everything mentioned (such as simplices, left Kan extensions, ...) internal.

Summary

1. Given an appropriate ∞-category with universes, we can construct an "algebra classifier" for finite algebraic data.
2. In particular we can get the "internal abelian group classifier" $\mathcal{U}_{\mathcal{A}\mathrm{b}}$ by adding conditions step by step.
3. The construction is functorial in the diagram, not yet in the limit shapes.

The End

Thanks for your attention! Questions?