

## Short-Term Scientific Mission Grant - APPLICATION FORM<sup>1</sup> -

Action number: CA20111

Applicant name: Danel Ahman

### Details of the STSM

Title: Comodule representations of second-order functionals

Start and end date: 09/09/2024 to 24/09/2024

### Goals of the STSM

Purpose and summary of the STSM.

1. Finalise a journal paper on comodule representations of second-order functionals, which resulted from the previous STSM that took place in the spring of 2024, and whose preliminary content was presented at TYPES 2024.
2. Extend the theory of comodule representations to incorporate notions of partial second-order functionals.
3. Give an abstract account of reductions in computability theory using comodule representations.

### Working Plan

Description of the work to be carried out by the applicant.

In a previous STSM, which took place in the spring of 2024, we developed a general notion of representations of second-order functionals based on comodules for monads on containers. This notion subsumes many kinds of representations of functionals, such as Brouwer's well-founded trees, finitely-supported functionals, instance reductions, and variations involving computational effects. The preliminary findings were presented at TYPES 2024 [1].

We have since started writing a journal paper that consolidates and expands on the results presented at TYPES 2024. The paper is in good shape and close to completion. Our first task will be to finalise it and make it ready for publication.

In addition, in the proposed STSM we propose two further venues of investigation:

1. **Partial functionals:** While comodule representations work well for various kinds of total second-order functionals, it is less clear how to get a fully satisfying account of representations of partial functionals. We shall investigate several variants, listed approximately in the order of technical difficulty: functionals whose partiality is induced solely by partiality of their arguments, functionals

<sup>1</sup> This form is part of the application for a grant to visit a host organisation located in a different country than the country of affiliation. It is submitted to the COST Action MC via e-COST. The Grant Awarding Coordinator coordinates the evaluation on behalf of the Action MC and informs the Grant Holder of the result of the evaluation for issuing the Grant Letter.

whose partiality can be expressed using the delay monad [2], and functionals whose partiality is (additionally) induced by representations by (coinductively defined) non-well-founded trees.

2. **Reductions in computability theory:** In computability theory, there are several notions of reductions, such as truth-table, many-to-one, and Turing reductions [3]. They can be construed as second-order functionals mapping oracles (infinite binary sequences) to oracles. Our initial investigations show that the various notions of reductions ought to correspond to comodule representations, but certain technical obstacles need to be overcome. We hope to get an abstract type-theoretic account, possibly using modal operators, which ought to complement and enrich recent work [4, 5] on type-theoretic accounts of computability theory. We shall look closely at how Turing degrees appear in realizability models (assemblies over the first Kleene algebra), and abstract the specific situation so that it becomes expressible in terms of comodule representations.

[1] Daniel Ahman, Andrej Bauer. Comodule Representations of Second-Order Functionals. 30th International Conference on Types for Proofs and Programs, TYPES 2024 (Copenhagen, June 2024)

[2] Venanzio Capretta. General recursion via coinductive types. Log. Methods Comput. Sci. 1(2), article 1 (2005)

[3] Hartley Rogers, Jr. Theory of Recursive Functions and Effective Computability (Reprint of the 1967 edition). MIT Press, 1987

[4] Haoyi Zeng, Yannick Forster and Dominik Kirst. Post's Problem and the Priority Method in CIC. 30th International Conference on Types for Proofs and Programs, TYPES 2024 (Copenhagen, June 2024)

[5] Andrew Sawn. Oracle modalities. 30th International Conference on Types for Proofs and Programs, TYPES 2024 (Copenhagen, June 2024)

### **Expected outputs and contribution to the Action MoU objectives and deliverables.**

Main expected results and their contribution to the progress towards the Action objectives (either research coordination and/or capacity building objectives) and deliverables.

The proposed research contributes most to the goals of WG6 (Type theory).

The results of the STSM will provide an abstract type-theoretic and category-theoretic account of several seemingly unrelated phenomena involving second-order functionals, ranging from continuity, uniform continuity, computational effects, to computability-theoretic reductions.

We also expect to gain new mathematical insights into the theory of containers from this research, especially into monads on containers. Containers play a well-known visible role in the type-theoretic and mathematical modelling of parametric data types and polymorphic functions, interacting systems, solutions to the view-update problem in database theory, gradient descent based machine learning algorithms, and many more fields. Our work will highlight their additional utility for the representation of second-order functionals and computability theory.

The proposed work on partial functionals and computability-theoretic reductions will provide the basis for suitably abstract approaches to these topics that side-steps many technical details, thus suggesting new approaches to formalisation of partiality and computability theory.

The results of the STSM will be presented in internationally recognized research venues, such as conferences and peer-reviewed journals.