Report on the outcomes of a Short-Term Scientific Mission[1]

Action number: CA20111

Grantee name: Danel Ahman

Details of the STSM

Title: Propositional containers and reductions in computability theory

Start and end date: 25/02/2024 to 09/03/2024

Description of the work carried out during the STSM

Description of the activities carried out during the STSM. Any deviations from the initial working plan shall also be described in this section.

(max. 500 words)

During the STSM I, together with my host Andrej Bauer, investigated the mathematical structures underpinning different forms of reduction studied in computability theory and reverse constructive mathematics, and related them to the notion of containers studied in type theory.

Building on the well-known observation (dating back to Brouwer) that the continuity of certain second-order functionals F can be captured by representing them with question-answer trees, we developed a novel abstract characterisation of what it means for a dependently typed second-order functional to be (tree) representable in such a way. Specifically, we observed that the tree representation of a functional F can be captured abstractly using the languages of category theory, type theory, and containers in terms of just three mathematically natural components: i) a monad T on the category of containers, describing the (tree) structure used in representations; ii) a comodule for the monad T, describing how the inputs of the given functional F act on its representation; and iii) a container morphism in the Kleisli category of the monad T, describing how for each instance of the functional F's output a representation is chosen and how F's value is computed from such representation. When a functional F is equal to the composition of the comodule structure map with the Kleisli map (under the functorial action of the carrier of the comodule structure), we say that F has a *comodule representation*. Such comodule representations compose as morphisms in the Kleisli category for the monad T, and thus form a category of their own.

By varying the monad T and the comodule structure, we could then capture many interesting situations in which comodule representations arise, in addition to continuity and the question-answer tree

[1] This report is submitted by the grantee to the Action MC for approval and for claiming payment of the awarded grant. The Grant Awarding Coordinator coordinates the evaluation of this report on behalf of the Action MC and instructs the GH for payment of the Grant.

representation that were our starting points. For instance, when using the finite powerset monad, the comodule representable functionals are those that have finite support—when restricting to propositional containers, these are those functionals that correspond to truth-table reductions. When using the inhabited powerset monad on propositional containers, the comodule representable functionals are those that correspond to instance reductions. When using the identity (resp. exception) monad, the comodule representable functionals are those that correspond to functional (resp. exceptional) reductions. Moreover, in addition to capturing existing notions of reduction from computability theory, the definition also allows us to explore what kinds of representations correspond to other well-known monads from computer science. Specifically, we looked at using the IO monad, in which case we capture functionals that compute values by additionally interacting with their environment (in a very controlled way).

Compared to the initial working plan, during the STSM we made the connection between containers and reductions more precise not by studying the further specifics of the category of propositional containers, but instead by developing and studying the novel notion of comodule representation of second-order functionals. As a result of developing and exploring a completely novel concept, the limited duration of the STSM did not give us enough time to also formalise our results and the associated mathematics in a proof assistant.

Description of the STSM main achievements and planned follow-up activities

Description and assessment of whether the STSM achieved its planned goals and expected outcomes, including specific contribution to Action objective and deliverables, or publications resulting from the STSM. Agreed plans for future follow-up collaborations shall also be described in this section.

(max. 500 words)

The main achievement of the STSM was to come up with the novel abstract definition of comodule representation of dependently typed second-order functionals, which distils the representation of functionals into a select few mathematically natural components (a monad, a comodule, and a Kleisli container map). The importance of this definition lies in its wide applicability—not only are we able to capture in a single unifying framework the tree-based representation of continuous functionals and a variety of notions of reduction appearing in computability theory, but it has also allowed us to explore other, novel forms of representation of functionals, such as, based on the functionals' interaction with an environment.

Furthermore, the notion of comodule representation developed during the STSM gives a fresh and more precise outlook (compared to our preliminary work preceding this STSM) on the relationship between (propositional) container data structures used in type theory and different forms of reduction appearing in computability theory—thus fulfilling the main goal of the STSM on more closely and precisely relating these two topics, and also contributing towards the Action's objectives by helping to facilitate transfer of knowledge between the two communities.

During the STSM, we submitted a two-page extended abstract based on the work done to the TYPES 2024 conference. We have also started writing up a journal article based on the work done during the STSM, and plan to continue collaborating on the topic matter in future. I also delivered a research seminar at the host institution during my visit detailing the findings of the STSM.