Report on the outcomes of a Short-Term Scientific Mission[1]

Action number: CA20111

Grantee name: Horatiu Cheval

Details of the STSM

Title: Proof mining libraries for Lean

Start and end date: 11/05/2025 to 17/05/2025

Description of the work carried out during the STSM

Description of the activities carried out during the STSM. Any deviations from the initial working plan shall also be described in this section.

(max. 500 words)

The objective of the STSM was to establish a collaboration between the host, Thomas Powell, and the grantee, Horatiu Cheval, on topics relating formalized mathematics (particularly in Lean) and proof mining. Two main directions are distinguished here: formalizing mathematical applications of proof mining in a library, and implementing some of the logical instruments used in practice to obtain such results, like Gödel's Dialectica interpretation.

We started by discussing in detail the previous Lean formalization of topics related to proof mining realized by the grantee. This previous work was also presented by the grantee in a talk given in the group's seminar.

A first project we agreed to work on is a Lean library of formalized convergence lemmas for sequences of real numbers satisfying certain recursive inequalities. This types of lemmas are essential in the proofs of a large number of results from optimization. Thus, such a library would facilitate the formalization of the convergence theorems for many optimization algorithms. Moreover, some of these lemmas had already been given quantitative counterparts in proof mining, allowing one to obtain rates for the algorithms, which we also want to capture in our library.

During the visit, we started writing this library, beginning with definitions of relevant quantitative notions like rates of convergence, and some basic proofs about them. We also spent time investigating what is

[1] This report is submitted by the grantee to the Action MC for approval and for claiming payment of the awarded grant. The Grant Awarding Coordinator coordinates the evaluation of this report on behalf of the Action MC and instructs the GH for payment of the Grant.

the most general setting in which they can be formulated, in order to closely follow mathlib's generality.

Regarding the second topic, we discussed different ways of encoding the Dialectica interpretation in Lean and their respective benefits and drawbacks: at the object level through an embedded logical system, or directly at the level of Lean expressions, possibly through metaprogramming. We decided that the second version would be better in terms of usability, and we investigated ways of expressing it using as little metaprogramming as possible. We concluded that an approach based on rewriting is promising, and decided to keep pursuing it going forward.

Description of the STSM main achievements and planned follow-up activities

Description and assessment of whether the STSM achieved its planned goals and expected outcomes, including specific contribution to Action objective and deliverables, or publications resulting from the STSM. Agreed plans for future follow-up collaborations shall also be described in this section.

(max. 500 words)

The STSM achieved its objectives in establishing a collaboration between the host, Thomas Powell, and the grantee, Horatiu Cheval, on the topics mentioned in the previous section. This relates to WG4: Libraries of formal proofs, and Capacity Building Objective 5: Actively support young researchers, the under-represented gender, and teams from regions with less capacity.

In the future, we plan:

• To continue developing our Lean library of quantitative convergence results for sequences of real numbers. We will extend it with various lemmas, proceeding with those surveyed in [1], expressed as generally as possible and with a user-friendly API. In the future, we also intend to build some automation that tries to match optimization algorithms with lemmas in our library, in order to partly automate their convergence proofs. This will be eventually be open-sourced.

• To investigate and implement ways of expressing Dialectica directly in Lean with as little metaprogramming as possible, following a rewriting approach. We identify both engineering and theoretical challenges (for example, the interaction between dependent types and the standard Dialectica interpretation) that need to be adressed. Nevertheless, we think that the approach can be implemented so that it is useful in many practical situations.

We expect publications to result from of both these research directions.

[1] B. Franci, S. Grammatico. "Convergence of sequences: A survey." *Annual Reviews in Control* 53 (2022): 161-186

Thomas Powell