Kripke-Joyal Forcing for Martin-Löf Type Theory

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Motivation

- Martin Löf type theory (MLTT) is common generalization of first-order logic (FOL) and the simply-typed lambda calculus, and is a powerful and expressive system of formal logic.
- It serves as the basis of Homotopy Type Theory, as well as several computer proof systems such as Agda, Coq, and Lean.
- It is a challenging problem to give semantics for MLTT that are both precise enough to strictly model the syntax and yet flexible enough to admit basic mathematical constructions.
- Kripke-Joyal forcing provides such semantics for both FOL and HOL and is here generalized to MLTT.

Kripke-Joyal forcing for FOL

Let $\mathbb{C}$ be a small category. For the topos of presheaves, write

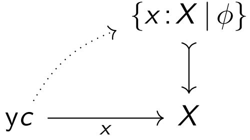
$$\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]\,.$$

We interpret a FOL formula $x\!:\!X \,|\, \phi$ over $X \in \widehat{\mathbb{C}}$ as a subobject,

$$\{x\!:\!X \,|\, \phi\} \rightarrowtail X\,.$$

Definition. Let $x : \mathsf{y}c \to X$. We say that x **forces** ϕ **at stage** c, if there is a factorization as on the right below.

$$c \Vdash \phi(x)$$



Kripke-Joyal forcing for FOL

Key fact: We can recursively unwind the condition $c \Vdash \phi(x)$ according to the structure of ϕ,

$$
\begin{array}{lll}
c \Vdash \phi(x) \vee \psi(x) & \text{iff} & c \Vdash \phi(x) \text{ or } c \Vdash \psi(x) \\
c \Vdash \phi(x) \wedge \psi(x) & \text{iff} & c \Vdash \phi(x) \text{ and } c \Vdash \psi(x) \\
c \Vdash \phi(x) \Rightarrow \psi(x) & \text{iff} & d \Vdash \phi(xf) \text{ implies } d \Vdash \psi(xf), \text{ for all } f : d \to c \\
c \Vdash \exists y.\vartheta(x, y) & \text{iff} & c \Vdash \vartheta(x, y) \text{ for some } y : \mathrm{y}c \to Y \\
c \Vdash \forall y.\vartheta(x, y) & \text{iff} & d \Vdash \vartheta(xf, y) \text{ for all } f : d \to c \text{ and } y : \mathrm{y}d \to Y
\end{array}
$$

This provides a way to determe whether ϕ **holds**, in the sense that

$$c \Vdash \phi(x) \qquad \text{for all } x : \mathrm{y}c \to X$$

which is equivalent to $\{x : X \mid \phi\} = X$.

Kripke-Joyal forcing for MLTT

For MLTT we instead need to force a **dependent type**

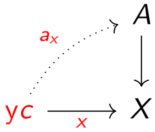
$$x : X \vdash A\,,$$

which is interpreted as a map $A \to X$ (an indexed family A_x), rather than a mere subobject $\{x : X \mid \phi\} \rightarrowtail X$.

This will require **forcing a term in context**,

$$c \Vdash a_x : A_x$$

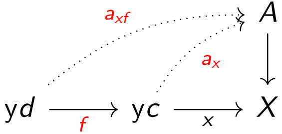
which is interpreted as a partial section.



Kripke-Joyal forcing for MLTT

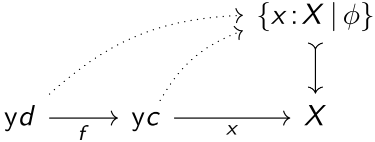
For this we need a **strict** interpretation:

$$\frac{c \Vdash a_x : A_x}{d \Vdash a_{xf} : A_{xf}}$$



unlike the propositional case:

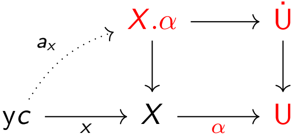
$$\frac{c \Vdash \phi(x)}{d \Vdash \phi(xf)}$$



Kripke-Joyal forcing for MLTT

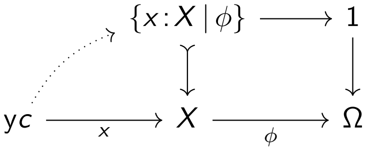
We use a **universe** to ensure coherence.

$c \Vdash a_x : \alpha(x)$



This is like using the **subobject classifier** to interpret FOL.

$c \Vdash \phi(x)$



Kripke-Joyal forcing for MLTT

Proposition (Forcing terms)

For any type in context $X \vdash \alpha$ *the following are equivalent.*

- *there is a term* t *such that*

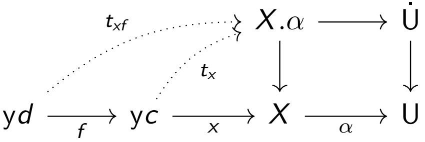
$$X \vdash t : \alpha$$

- *for all* $x : \mathrm{y}c \to X$ *there is given* **coherently** t_x *such that*

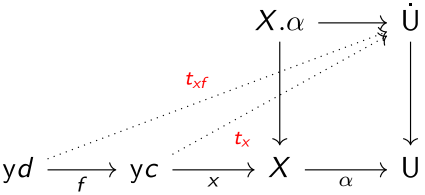
$$c \Vdash t_x : \alpha(x)\,.$$

Kripke-Joyal forcing for MLTT

Proof. Coherence means that $t_{xf} = t_x \circ f$.



But these partial sections correspond to partial lifts of α,



Kripke-Joyal forcing for MLTT

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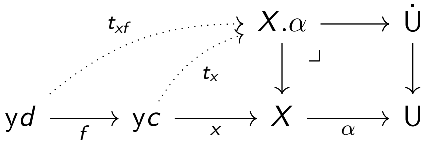

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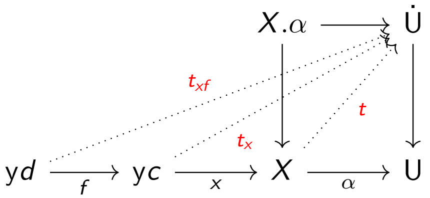


So $X \vdash t : \alpha$ by Yoneda.

The natural model of MLTT

Let $f : Y \to X$ and consider the two-pullbacks diagram arising from substitution.

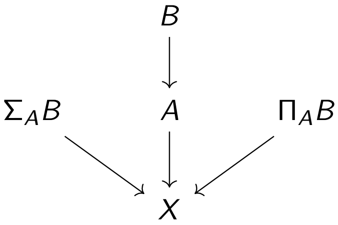
$$\frac{X \vdash \alpha}{Y \vdash \alpha f} \qquad \begin{array}{ccccc} Y.\alpha f & \longrightarrow & X.\alpha & \longrightarrow & \dot{\mathsf{U}} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow[f]{} & X & \xrightarrow[\alpha]{} & \mathsf{U} \end{array}$$

The pullback functor $f^* : \mathcal{E}/_X \to \mathcal{E}/_Y$ is thus modeled by precomposition.

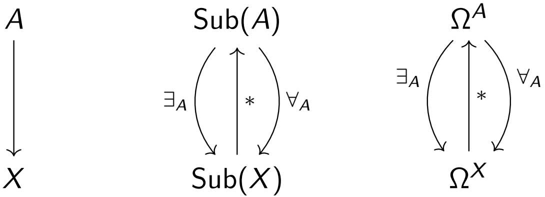
$$\begin{array}{c} Y \\ \downarrow{\scriptstyle f} \\ X \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}(Y, \mathsf{U}) & \hookrightarrow & \mathcal{E}/_Y \\ {\scriptstyle -\circ f}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{Hom}(X, \mathsf{U}) & \hookrightarrow & \mathcal{E}/_X \end{array}$$

The natural model of MLTT

For small $A \to X$ the adjoint functors $\Sigma_A B \dashv A^* \dashv \Pi_A B$



are induced by a structure on $\dot{\mathsf{U}} \to \mathsf{U}$, just as $\forall_A$ and $\exists_A$ are induced by maps on powerobjects.



2. The natural model of MLTT

Indeed, let $PX = \sum_{A:\mathsf{U}} X^{[A]}$, then we have:

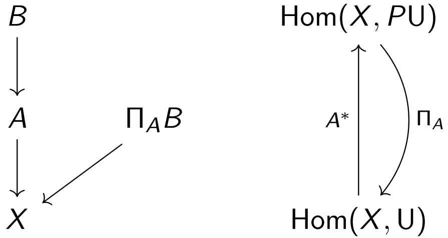
Proposition (A 2017)

The universe $\dot{\mathsf{U}} \to \mathsf{U}$ *models the rules for products just if there are maps* λ *and* Π *making a pullback diagram.*

$$
\begin{array}{ccc}
P\dot{\mathsf{U}} & \xrightarrow{\quad\lambda\quad} & \dot{\mathsf{U}} \\
\big\downarrow & \lrcorner & \big\downarrow \\
P\mathsf{U} & \xrightarrow[\quad\Pi\quad]{} & \mathsf{U}
\end{array}
$$

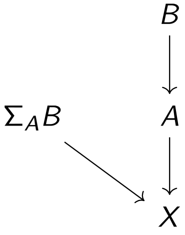
The natural model of MLTT

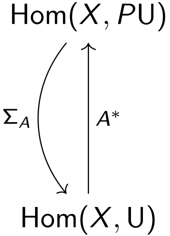
The right adjoint $A^* \dashv \Pi_A B$ is then induced by composing the classifying map with $\Pi : P\mathsf{U} \to \mathsf{U}$.



The natural model of MLTT

A similar structure $\Sigma : P\mathsf{U} \to \mathsf{U}$ induces the left adjoint $\Sigma_A \dashv A^*$.





The natural model of MLTT

Proposition

The natural model structure on the universe,

$$\begin{array}{ccc} Q & \xrightarrow{\text{pair}} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow \\ P\mathsf{U} & \xrightarrow[\Sigma]{} & \mathsf{U} \end{array} \qquad \begin{array}{ccc} P\dot{\mathsf{U}} & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow \\ P\mathsf{U} & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

provides a **strict** interpretation of MLTT, permitting forcing conditions for Σ and Π in the form

$$c \Vdash t : \Sigma_{y:\alpha(x)}\beta(x,y)$$

$$c \Vdash t : \Pi_{y:\alpha(x)}\beta(x,y)$$

The Kripke-Joyal forcing rules

Theorem (AGH 2022)

Let $X \in \widehat{\mathbb{C}}$ *and* $\alpha : X \to \mathsf{U}$ *and* $\beta : X.\alpha \to \mathsf{U}$*.*
For all $x : \mathsf{y}c \to X$*, we have*

$$
\begin{array}{lll}
c \Vdash \ t : 0 & \textit{iff} & t \neq t \\
c \Vdash \ t : 1 & \textit{iff} & t = * \\
c \Vdash \ t : (\alpha + \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{or} \quad c \Vdash b : \beta(x) \\
c \Vdash \ t : (\alpha \times \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{and} \quad c \Vdash b : \beta(x) \\
c \Vdash \ t : (\Sigma_\alpha \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{and} \quad c \Vdash b : \beta(x, a) \\
c \Vdash \ t : (\Pi_\alpha \beta)(x) & \textit{iff} & \textit{for all } f : d \to c \textit{ and } d \Vdash a : \alpha(xf) \\
 & & \textit{there's } d \Vdash b_{f,a} : \beta(xf, a) \textit{ coherently}
\end{array}
$$

The completeness theorem

Say $\mathbb{C}$ **forces a term of type** α,

$$\mathbb{C} \Vdash X \vdash t : \alpha\,,$$

if for all $c \in \mathbb{C}$ and all $x : \mathrm{y}c \to X$, there is given coherently

$$c \Vdash t : \alpha(x)\,.$$

Theorem (AGH 2022)

Let α *be a closed type in* MLTT *with the type forming operations*

$$1,\ X,\ A \times B,\ A \to B,\ \Sigma_A B,\ \Pi_A B,\ s =_A t\,.$$

There is a closed term $\vdash t : \alpha$ *if, and only if, for all categories* $\mathbb{C}$ *and all presheaves* X *on* $\mathbb{C}$*, one has* $\mathbb{C} \Vdash t : \alpha$*. Briefly,*

$$\mathsf{MLTT} \vdash t : \alpha \qquad \textit{iff} \qquad \mathbb{C} \Vdash t : \alpha \quad \textit{for all } \mathbb{C} \textit{ and } X.$$

Moreover, it suffices to assume that $\mathbb{C}$ *is a poset.*

References

1. Awodey, S. (2017) Natural models of homotopy type theory, Mathematical Structures in Computer Science, 28(2).
2. Awodey, S. and N. Gambino and S. Hazratpour (2022) Kripke-Joyal forcing for homotopy type theory and uniform fibrations, arXiv:2110.14576.

The completeness theorem

Proof. Let $P = \mathcal{O}X_{\mathbb{T}}$, where $\mathbb{T}$ is the classifying category of MLTT, and $p : \mathrm{Sh}(X_{\mathbb{T}}) \twoheadrightarrow \widehat{\mathbb{T}}$ is the spatial cover.

There are LCCC embeddings:

$$\mathbb{T} \xhookrightarrow{\mathsf{y}} \widehat{\mathbb{T}} \xhookrightarrow{p^*} \mathrm{Sh}(X_{\mathbb{T}}) \longrightarrow \widehat{\mathcal{O}X_{\mathbb{T}}}.$$

So we have:

$$\begin{aligned}
\mathrm{MLTT} \vdash t : C &\iff 1 \xrightarrow{t} C && \mathbb{T} \\
&\iff 1 \cong \mathsf{y}1 \xrightarrow{\mathsf{y}t} \mathsf{y}C \cong [\![C]\!]^{\mathbb{T}} && \widehat{\mathbb{T}} \\
&\iff 1 \cong p^*\mathsf{y}1 \xrightarrow{p^*\mathsf{y}t} p^*\mathsf{y}C \cong [\![C]\!]^{X_{\mathbb{T}}} && \mathrm{Sh}(X_{\mathbb{T}}) \\
&\iff \mathcal{O}X_{\mathbb{T}} \Vdash t : C && \square
\end{aligned}$$