B-systems and C-systems are equivalent

Jacopo Emmenegger
Università degli Studi di Genova

joint work with Benedikt Ahrens, Paige Randall North
and Egbert Rijke

Workshop on Syntax and Semantics of Type Theory
Stockholms universitet
20–21 May 2022

C-systems (a.k.a. contextual categories[1])

A category $\mathcal{C}$ with a terminal object 1 together with structure such that:

1. the objects of $\mathcal{C}$ can be arranged into a rooted tree $R(\mathcal{C})$ (with root 1) that embeds into $\mathcal{C}$, and
2. every cospan in $\mathcal{C}$ can be completed to a pullback if one of the arrows, say p, is coming from $R(\mathcal{C})$. This choice is functorial in the other arrow, and such that the chosen pullback of p is also coming from $R(\mathcal{C})$.

[1] Cartmell. Generalised algebraic theories and contextual categories. 1986.

C-systems (a.k.a. contextual categories[1])

A category $\mathcal{C}$ with a terminal object 1 together with structure such that:

1. the objects of $\mathcal{C}$ can be arranged into a rooted tree $R(\mathcal{C})$ (with root 1) that embeds into $\mathcal{C}$, and
2. every cospan in $\mathcal{C}$ can be completed to a pullback if one of the arrows, say p, is coming from $R(\mathcal{C})$. This choice is functorial in the other arrow, and such that the chosen pullback of p is also coming from $R(\mathcal{C})$.

Think of:

1. The category of contexts of a dependent type theory. Here display maps $(\Gamma, x : A) \to \Gamma$ are functorially stable under pullback.
2. A category with display maps à la Taylor $(\mathcal{C}, \mathcal{D})$, where the class of display maps $\mathcal{D}$ form a rooted tree and its stability under pullback is functorial. N.B. $\mathcal{D}$ is *not* required to be closed under composition.

[1] Cartmell. Generalised algebraic theories and contextual categories. 1986.

B-systems[2]

A B-frame $\mathbb{B}$:

$$
\begin{tikzcd}
 & & \tilde{B}_1 \arrow[dl, "\partial"'] & & \tilde{B}_2 \arrow[dl, "\partial"'] & & \tilde{B}_3 & \cdots \\
B_0 & B_1 \arrow[l, "\mathrm{ft}"'] & & B_2 \arrow[ll, "\mathrm{ft}"'] & & B_3 \arrow[ll, "\mathrm{ft}"'] & \cdots
\end{tikzcd}
$$

think of:

$A \in B_{n+1}$ as $v_1 : \mathrm{ft}^n(A), \ldots, v_n : \mathrm{ft}(A) \vdash A \text{ type}$

$a \in \tilde{B}_{n+1}$ as $v_1 : \mathrm{ft}^n(A), \ldots, v_n : \mathrm{ft}(A) \vdash a : A,$ where $A :\equiv \partial(a)$

[2]Voevodsky. B-systems. 2014

B-systems[2]

A B-frame $\mathbb{B}$:

$$\begin{tikzcd} & & \tilde{B}_1 \arrow[dl, "\partial"'] & & \tilde{B}_2 \arrow[dl, "\partial"'] & & \tilde{B}_3 \arrow[dl, "\partial"'] & \cdots \\ \{*\} \cong B_0 & B_1 \arrow[l, "\mathrm{ft}"'] & & B_2 \arrow[ll, "\mathrm{ft}"'] & & B_3 \arrow[ll, "\mathrm{ft}"'] & \cdots \end{tikzcd}$$

think of:

$A \in B_{n+1}$ as $v_1 : \mathrm{ft}^n(A), \ldots, v_n : \mathrm{ft}(A) \vdash A \ \mathrm{type}$

$a \in \tilde{B}_{n+1}$ as $v_1 : \mathrm{ft}^n(A), \ldots, v_n : \mathrm{ft}(A) \vdash a : A$, where $A := \partial(a)$

[2]Voevodsky. B-systems. 2014

B-systems[2]

A B-frame $\mathbb{B}$:

$$
\begin{tikzcd}
 & & \tilde{B}_1 \arrow[dl, "\partial"'] & & \tilde{B}_2 \arrow[dl, "\partial"'] & & \tilde{B}_3 \arrow[dl, "\partial"'] & \cdots \\
\{*\} \cong B_0 & B_1 \arrow[l, "\mathrm{ft}"'] & & B_2 \arrow[ll, "\mathrm{ft}"'] & & B_3 \arrow[ll, "\mathrm{ft}"'] & \cdots
\end{tikzcd}
$$

think of:

$A \in B_{n+1}$ as $v_1 : \mathrm{ft}^n(A), \ldots, v_n : \mathrm{ft}(A) \vdash A \text{ type}$

$a \in \tilde{B}_{n+1}$ as $v_1 : \mathrm{ft}^n(A), \ldots, v_n : \mathrm{ft}(A) \vdash a : A$, where $A := \partial(a)$

together with, for $x \in \tilde{B}_{n+1}$, $X \in B_n$, homomorphisms of B-frames

$$\mathbb{B}/\partial(x) \xrightarrow{S_x} \mathbb{B}/\mathrm{ft}\partial(x) \qquad \mathbb{B}/\mathrm{ft}(X) \xrightarrow{W_X} \mathbb{B}/X$$

and a function $\delta_n \colon B_{n+1} \to \tilde{B}_{n+2}$ for every n.

Plus a number of equations.

[2]Voevodsky. B-systems. 2014

Why B-systems?

Why B-systems?

In order to construct the initial model for a dependent type theory à la Martin-Löf:

1. From a suitable signature construct a monad of preterms $R\colon \mathbf{Set} \to \mathbf{Set}$ and a left module of pretypes $LM\colon \mathbf{Set} \to \mathbf{Set}$ over R.[3]
2. Construct a C-system $C(R, LM)$ of preterms and pretypes.[2]
3. Use subsystems and quotients of C-systems[4] to carve out well-formed types and well-typed terms modulo a convertibility relation.

It resembles usual presentations of Martin-Löf type theory, but:

- it is done for an arbitrary signature,
- it directly produces a C-system.

[3] Voevodsky. C-system of a module over a Jf-relative monad. 2016.
[4] Voevodsky. Subsystems and regular quotients of C-systems. 2016.

Why B-systems?

To address step 3 (carving out well-formed types and terms) need to develop a theory of subsystems and quotients of C-systems.

Use B-systems for this: there is a bijection[5] between

- subsystems of a C-system $\mathbb{C}$, and
- subsystems of a certain B-system $B(\mathbb{C})$, *i.e.* subsets of the B_n's and $\tilde{B}_{n+1}$'s closed under substitution, weakening and generic elements.

and similarly for quotients.

B-systems are algebras for a monad on a category of presheaves.[6]

[5]Voevodsky. Subsystems and regular quotients of C-systems. 2016.

[6]Garner. Combinatorial structure of type dependency. 2014.

Why B-systems?

To address step 3 (carving out well-formed types and terms) need to develop a theory of subsystems and quotients of C-systems.

Use B-systems for this: there is a bijection[5] between

- subsystems of a C-system $\mathbb{C}$, and
- subsystems of a certain B-system $B(\mathbb{C})$, *i.e.* subsets of the B_n's and $\tilde{B}_{n+1}$'s closed under substitution, weakening and generic elements.

and similarly for quotients.

B-systems are algebras for a monad on a category of presheaves.[6]

Conjecture (Voevodsky)

The categories of B-systems and C-systems are equivalent.

[5] Voevodsky. Subsystems and regular quotients of C-systems. 2016.

[6] Garner. Combinatorial structure of type dependency. 2014.

Why B-systems?

To address step 3 (carving out well-formed types and terms) need to develop a theory of subsystems and quotients of C-systems.

Use B-systems for this: there is a bijection[5] between

- subsystems of a C-system $\mathbb{C}$, and
- subsystems of a certain B-system $B(\mathbb{C})$, *i.e.* subsets of the B_n's and $\tilde{B}_{n+1}$'s closed under substitution, weakening and generic elements.

and similarly for quotients.

B-systems are algebras for a monad on a category of presheaves.[6]

Theorem (Ahrens–E.–North–Rijke)

The categories of B-systems and C-systems are equivalent.

[5] Voevodsky. Subsystems and regular quotients of C-systems. 2016.

[6] Garner. Combinatorial structure of type dependency. 2014.

B-systems: equations

1. Substitution distributes over substitution and over weakening:

$$S_x S_y = S_{S_x(y)} S_x \qquad S_x W_X = W_{S_x(X)} S_x$$

   where $x \in \tilde{B}_{n+1}$ and $y \in \tilde{B}_{m+n+1}$ s.t. $\mathrm{ft}^m \partial(y) = \partial(x)$, and it preserves generic elements

$$S_x \delta = \delta S_x$$

2. Similarly for weakening.

$$W_X S_y = S_{W_X(y)} W_X \quad W_X W_Y = W_{W_X(Y)} W_X \quad W_X \delta_n = \delta_{n+1} W_X$$

3. For $x \in \tilde{B}_{n+1}$ and $X \in B_{n+1}$:

$$S_x W_{\partial(x)} = \mathrm{id}_{\mathbb{B}/\partial(x)} \qquad S_x(\delta\partial(x)) = x \qquad S_{\delta(X)}(W_X/X) = \mathrm{id}_{\mathbb{B}/X}$$

B-systems: rooted tree

$$\{*\} \longleftarrow B_1 \xleftarrow{\mathrm{ft}} B_2 \xleftarrow{\mathrm{ft}} B_3 \quad \cdots$$

is a rooted tree $\mathbf{T}(\mathbb{B})$ with graph given by:

$$V := \coprod_n B_n$$

$$(m, X) \to (n, Y) \quad \text{iff} \quad m = n + 1 \text{ and } Y = \mathrm{ft}(X).$$

$\mathbf{T}(\mathbb{B})$ encodes one-step type dependencies.

Overview

The functor **Bsys** $\to$ **Csys** will construct the "syntactic category" of a B-system.

1. Define contexts and context morphisms (i.e. tuples of well-typed terms).
2. Define composition of context morphisms by means of substitution.

Overview

The functor **Bsys** $\to$ **Csys** will construct the "syntactic category" of a B-system.

1. Define contexts and context morphisms (i.e. tuples of well-typed terms).
2. Define composition of context morphisms by means of substitution.

$$
\begin{tikzcd}
\mathbf{Esys} \arrow[rrr, shift right=1.5] \arrow[rrr, phantom, "\simeq"] & & & \mathbf{rCEsys} \arrow[lll, shift right=1.5] \\
\mathbf{Bsys} \arrow[u, hook] \arrow[r, "\simeq"] & \overset{\textbf{stratified}}{\mathbf{Esys}} \arrow[ul, hook] \arrow[r, phantom, "\simeq"] & \overset{\textbf{stratified}}{\mathbf{rCEsys}} \arrow[ur, hook] & \mathbf{Csys} \arrow[u, hook] \arrow[l, "\simeq"']
\end{tikzcd}
$$

In the top row, replace the rooted tree of single-step type dependencies with an arbitrary (strict) category of multi-step type dependencies.

Stratification

A (strict) category with terminal object 1 is stratified if there is a functor $L\colon \mathcal{C} \to (\mathbb{N}, \geq)$ such that

1. $L(X) = 0$ if and only if X is the chosen terminal object 1,
2. for any $f : X \to Y$ we have $L(X) = L(Y)$ if and only if $X = Y$ and $f = \mathrm{id}_X$, and
3. every morphism $f : X \to Y$ in $\mathcal{C}$, where $L(X) = n + m + 1$ and $L(Y) = n$, has a unique factorization

$$X = X_{m+1} \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = Y$$

where $L(X_{i+1}) = n + i + 1 = L(X_i) + 1$.

Proposition

1. Being stratified is a property.

Stratification

A (strict) category with terminal object 1 is stratified if there is a functor $L\colon \mathcal{C} \to (\mathbb{N}, \geq)$ such that

1. $L(X) = 0$ if and only if X is the chosen terminal object 1,
2. for any $f : X \to Y$ we have $L(X) = L(Y)$ if and only if $X = Y$ and $f = \mathrm{id}_X$, and
3. every morphism $f : X \to Y$ in $\mathcal{C}$, where $L(X) = n + m + 1$ and $L(Y) = n$, has a unique factorization

$$X = X_{m+1} \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = Y$$

   where $L(X_{i+1}) = n + i + 1 = L(X_i) + 1$.

Proposition

1. Being stratified is a property.
2. The category of stratified categories is equivalent to the category of rooted trees via the free category functor.

CE-systems

A CE-system consists of

1. two strict category structures $\mathcal{F}$ and $\mathcal{C}$ on the same set of objects $\mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{C})$,
2. an identity-on-objects functor $I \colon \mathcal{F} \to \mathcal{C}$ between them,
3. a chosen object $\top$ which is terminal in $\mathcal{F}$, and
4. for every $f \colon \Delta \to \Gamma$ in $\mathcal{C}$ and any $A \in \mathcal{F}/\Gamma$, a choice of a pullback square

$$\begin{tikzcd}
\Delta.f^*A \arrow[r, "\pi_2(f,A)"] \arrow[d, two heads, "I(f^*A)"'] & \Gamma.A \arrow[d, two heads, "I(A)"] \\
\Delta \arrow[r, "f"] & \Gamma
\end{tikzcd}$$

which is functorial in A and f.

A CE-system is rooted if $I(\top) = \top$ is terminal in $\mathcal{C}$.

$$
\begin{tikzcd}
\textbf{Esys} \arrow[rrr, shift right, "\simeq"'] & & & \textbf{rCEsys} \arrow[lll, shift right] \\
\textbf{Bsys} \arrow[u, hook] \arrow[r, "\simeq"] & \overset{\textbf{stratified}}{\textbf{Esys}} \arrow[ul, hook] \arrow[r, phantom, "\simeq"] & \overset{\textbf{stratified}}{\textbf{rCEsys}} \arrow[ur, hook] & \textbf{Csys} \arrow[l, "\simeq"'] \arrow[u, hook]
\end{tikzcd}
$$

C-system:

- Functorial choice of pullbacks.
- Rooted tree of type dependency.

CE-system:

- Functorial choice of pullbacks.
- If stratified: rooted tree of type dependency.

E-systems

An E-system $\mathbb{E}$ consists of a category $\mathcal{F}$ with a terminal object $[]$ together with:

1. A term structure: for every $A \in \mathcal{F}/\Gamma$, a set $T(A)$.
2. A substitution structure: for every $A \in \mathcal{F}/\Gamma$ and $x \in T(A)$, a functor
$$S_x \colon \mathcal{F}/\Gamma.A \to \mathcal{F}/\Gamma$$
with term structure: for every $B \in \mathcal{F}/\Gamma.A$, a function
$$T(B) \to T(S_x(B)).$$
3. A weakening structure: for every $A \in \mathcal{F}/\Gamma$, a functor with term structure $W_A \colon \mathbb{E}/\Gamma \to \mathbb{E}/\Gamma.A$.
4. A projection structure: for every $A \in \mathcal{F}/\Gamma$, an element $\mathrm{idtm}_A \in T(W_A(A))$.

Satisfying equations similar to those of B-systems, plus:
S_x and W_A preserve terminal objects, and
W_A is functorial in A, *i.e.* $W_{\mathrm{id}_\Gamma} = \mathrm{Id}_{\mathbb{E}/\Gamma}$ and $W_{AP} = W_P W_A$.

From B-systems to E-systems

$\mathbb{B}$ a B-system. $\mathcal{F}$ is the free category on the (graph of the) rooted tree $\mathbf{T}(\mathbb{B})$ of single-step type dependencies.

Arrows in $\mathcal{F}$ are of the form $(X, k)\colon (n+k, X) \to (n, \mathrm{ft}^k(X))$.

From B-systems to E-systems

$\mathbb{B}$ a B-system. $\mathcal{F}$ is the free category on the (graph of the) rooted tree $\mathbf{T}(\mathbb{B})$ of single-step type dependencies.

Arrows in $\mathcal{F}$ are of the form $(X,k)\colon (n+k, X) \to (n, \mathrm{ft}^k(X))$.

The term structure $T(X,k)$ is defined inductively together with simultaneous substitutions S^k_t of tuples of terms $t \in T(X,k)$.

- On arrows of length 0 and 1: for $X \in B_n$

$$T(X,0) := \{*\} \qquad S^0_* := \mathrm{id}\colon \mathbb{B}/X \to \mathbb{B}/X$$
$$T(X,1) := \partial^{-1}(X) \subseteq \tilde{B}_n \qquad S^1_x := S_x\colon \mathbb{B}/X \to \mathbb{B}/\mathrm{ft}(X)$$

- For $X \in B_n$ and $k \leq n$ by induction:
  suppose that for every $m \leq n$ and $Y \in B_m$ we have $T(Y,k)$ for $k \leq m$ and $S^k_s\colon \mathbb{B}/Y \to \mathbb{B}/\mathrm{ft}^k(Y)$ for $s \in T(Y,k)$, then

$$T(X,k{+}1) := \coprod_{t \in T(\mathrm{ft}(X),k)} T(S^k_t(X),1) \qquad S^{k+1}_{(t,x)} := S_x(S^k_t/X)$$

$$
\begin{tikzcd}
\textbf{Esys} \arrow[rrr, shift right, "\cong"'] & & & \textbf{rCEsys} \arrow[lll, shift right] \\
\textbf{Bsys} \arrow[u, hook, red] \arrow[r, red, "\simeq"] & \overset{\textbf{stratified}}{\textbf{Esys}} \arrow[ul, hook, red] \arrow[r, phantom, "\simeq"] & \overset{\textbf{stratified}}{\textbf{rCEsys}} \arrow[ur, hook] & \textbf{Csys} \arrow[u, hook] \arrow[l, "\simeq"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\textbf{Esys} \arrow[rrr, shift right, "\cong"'] & & & \textbf{rCEsys} \arrow[lll, shift right] \\
\textbf{Bsys} \arrow[u, hook] \arrow[r, "\simeq"] & \begin{array}{c}\textbf{stratified}\\ \textbf{Esys}\end{array} \arrow[ul, hook] \arrow[r, phantom, "\simeq"] & \begin{array}{c}\textbf{stratified}\\ \textbf{rCEsys}\end{array} \arrow[ur, hook] & \textbf{Csys} \arrow[u, hook] \arrow[l, "\simeq"']
\end{tikzcd}
$$

From CE-systems to E-systems

$I \colon \mathcal{F} \to \mathcal{C}$ a CE-system.
Define an E-system on $\mathcal{F}$ as follows:

$$T(A) \coloneqq \{x \colon \Gamma \to \Gamma.A \mid I(A)x = \mathrm{id}_\Gamma\}, \qquad \text{for } A \in \mathcal{F}/\Gamma$$

$$W_A \coloneqq A^* \colon \mathcal{F}/\Gamma \to \mathcal{F}/\Gamma.A, \qquad \text{for } A \in \mathcal{F}/\Gamma$$

$$S_x \coloneqq x^* : \mathcal{F}/\Gamma.A \to \mathcal{F}/\Gamma, \qquad \text{for } x \in T(A)$$

$$\begin{tikzcd}
\Gamma.A \arrow[dr, dotted, "\mathrm{idtm}_A"] \arrow[drr, bend left, "\mathrm{id}_{\Gamma.A}"] \arrow[ddr, bend right, "\mathrm{id}_{\Gamma.A}"'] & & \\
& \Gamma.A.W_A(A) \arrow[r, "\pi_2(I(A),A)"'] \arrow[d, "I(W_A(A))"] & \Gamma.A \arrow[d, "I(A)"] \\
& \Gamma.A \arrow[r, "I(A)"'] & \Gamma
\end{tikzcd}$$

From E-systems to CE-systems

$\mathbb{E}$ an E-system.

For $A, B \in \mathcal{F}/\Gamma$, an internal morphism over Γ is an element of

$$\mathrm{thom}(A, B) := T(W_A(B))$$

From E-systems to CE-systems

$\mathbb{E}$ an E-system.

For $A, B \in \mathcal{F}/\Gamma$, an internal morphism over Γ is an element of

$$\mathrm{thom}(A, B) := T(W_A(B))$$

Note that, given also $P \in \mathcal{F}/\Gamma.A$:

$$\mathrm{thom}(PA, B) = T(W_{PA}(B)) = T(W_P W_A(B)) = \mathrm{thom}(P, W_A(B))$$

Indeed once we have a category of internal morphisms

$$(-) \circ A \dashv W_A$$

From E-systems to CE-systems

$\mathbb{E}$ an E-system.

For $A, B \in \mathcal{F}/\Gamma$, an internal morphism over Γ is an element of

$$\mathrm{thom}(A, B) := T(W_A(B))$$

Given $f \in \mathrm{thom}(A, B)$, define precomposition by f as

$$\begin{tikzcd}
\mathbb{E}/\Gamma.B \arrow[rr, "f^*"] \arrow[dr, "W_A/B"'] & & \mathbb{E}/\Gamma.A \\
& \mathbb{E}/\Gamma.A.W_A(B) \arrow[ur, "S_f"'] &
\end{tikzcd}$$

Then if $g \in \mathrm{thom}(B, C)$, define $g \circ f := f^*(g)$.

Identities are given by $\mathsf{idtm}_A \in T(W_A(A)) = \mathrm{thom}(A, A)$.

Obtain a category $\mathcal{C}_{\mathbb{E}}(\Gamma)$ of internal morphisms over Γ.

From E-systems to CE-systems

Define $I_{\mathbb{E}}^{\Gamma} \colon \mathcal{F}/\Gamma \to \mathcal{C}_{\mathbb{E}}(\Gamma)$ as

$$
\begin{tikzcd}
\Gamma.A.P \arrow[rr, "P"] \arrow[dr, "PA"'] & & \Gamma.A \arrow[dl, "A"] \\
& \Gamma &
\end{tikzcd}
\longmapsto W_P(\mathsf{idtm}_A) \in \mathrm{thom}(PA, A)
$$

Proposition

$\mathbb{E}$ and E-system. For every Γ,

$$\mathcal{F}/\Gamma \xrightarrow{I_{\mathbb{E}}^{\Gamma}} \mathcal{C}_{\mathbb{E}}(\Gamma)$$

is a rooted CE-system.

The choice of pullbacks is also given algebraically using the operations of $\mathbb{E}$.

Equivalence

Theorem (Ahrens–E.–North–Rijke)

1. The functors **Esys** $\rightleftarrows$ **rCEsys** give rise to an equivalence of categories.

$$
\begin{tikzcd}
\textbf{Esys} \arrow[rrr, shift right, red] \arrow[rrr, phantom, "\simeq", red] & & & \textbf{rCEsys} \arrow[lll, shift right, red] \\
\textbf{Bsys} \arrow[u, hook] \arrow[r, "\simeq"] & \overset{\textbf{stratified}}{\textbf{Esys}} \arrow[ul, hook] \arrow[r, phantom, "\simeq"] & \overset{\textbf{stratified}}{\textbf{rCEsys}} \arrow[ur, hook] & \textbf{Csys} \arrow[u, hook] \arrow[l, "\simeq"']
\end{tikzcd}
$$

Equivalence

Theorem (Ahrens–E.–North–Rijke)

1. The functors **Esys** $\rightleftarrows$ **rCEsys** give rise to an equivalence of categories.
2. The equivalence restricts to stratified systems.

$$
\begin{tikzcd}
\mathbf{Esys} \arrow[rrr, shift right, "\simeq"'] & & & \mathbf{rCEsys} \arrow[lll, shift right] \\
\mathbf{Bsys} \arrow[u, hook] \arrow[r, "\simeq"] & \overset{\textbf{stratified}}{\mathbf{Esys}} \arrow[ul, hook] \arrow[r, phantom, "\simeq"] & \overset{\textbf{stratified}}{\mathbf{rCEsys}} \arrow[ur, hook] & \mathbf{Csys} \arrow[u, hook] \arrow[l, "\simeq"']
\end{tikzcd}
$$

Equivalence

Theorem (Ahrens–E.–North–Rijke)

1. The functors **Esys** $\rightleftarrows$ **rCEsys** give rise to an equivalence of categories.
2. The equivalence restricts to stratified systems.
3. It follows that **Bsys** $\equiv$ **Csys**.

$$
\begin{tikzcd}
\mathbf{Esys} \arrow[rrr, shift right, "\simeq"] & & & \mathbf{rCEsys} \arrow[lll, shift right] \\
\mathbf{Bsys} \arrow[u, hook] \arrow[r, "\simeq"] & \overset{\textbf{stratified}}{\mathbf{Esys}} \arrow[ul, hook] \arrow[r, phantom, "\simeq"] & \overset{\textbf{stratified}}{\mathbf{rCEsys}} \arrow[ur, hook] & \mathbf{Csys} \arrow[u, hook] \arrow[l, "\simeq"']
\end{tikzcd}
$$

Details in:
B-systems and C-systems are equivalent. `arXiv:2111.09948`