Characterizing clan-algebraic categories

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Duality for finite-limit theories (Gabriel-Ulmer duality[3])

Theorem

There is a bi-equivalence of 2-categories

$$\mathbf{FL} \quad \underset{\mathcal{L} \mapsto \mathbf{Mod}(\mathcal{L}) := \mathbf{FL}(\mathcal{L},\mathbf{Set})}{\overset{\{\text{compact objects}\}^{\mathrm{op}} \leftarrow\!\shortmid \mathfrak{X}}{\longleftrightarrow}} \quad \mathbf{LFP}^{\mathrm{op}}.$$

- **FL** is the 2-category of **small finite-limit categories** and finite-limit preserving functors
- **LFP** is the 2-category of **locally finitely presentable categories**, i.e.
  - locally small cocomplete categories with a dense set of compact (finitely presentable) objects, and
  - functors preserving small limits and filtered colimits ('forgetful functors').
- Intuition: view small lex categories as **theories**, and LFP categories as categories of models
- This makes sense since every lex category can be exhibited as categorical incarnation of an **essentially algebraic theory**[1] or a **generalized algebraic theory**[2]

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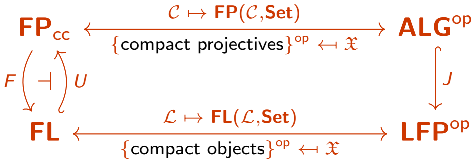
1 P. Freyd. "Aspects of topoi". In: *Bulletin of the Australian Mathematical Society* (1972).

2 J. Cartmell. "Generalised algebraic theories and contextual categories". In: *Annals of Pure and Applied Logic* (1986).

3 P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, 1971.

Duality for finite-product theories[4]

There's a 'restriction' of G–U duality to **finite-product theories** (corresponding to many-sorted **ordinary algebraic theories**):



- $\mathbf{FP}_{cc}$ is the 2-category of Cauchy-complete finite-product categories
- **ALG** is the 2-category of **algebraic categories** and **algebraic functors**
  - An **algebraic category** is an l.f.p. category which is Barr-exact and where the compact (regular) projective objects are dense
  - An **algebraic functor** is a functor that preserves small limits, filtered colimits, and regular epimorphisms.

  sifted colimits

- Clan-duality can be viewed as a **refinement** of GU-duality which allows to control the amount of limit-preservation in the models

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[4] J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press, 2010.

Clans

Definition

A **clan** is a small category $\mathcal{T}$ with terminal object 1, equipped with a class $\mathcal{T}_\dagger \subseteq \mathrm{mor}(\mathcal{T})$ of morphisms – called **display maps** and written $\twoheadrightarrow$ – such that

1. pullbacks of display maps along all maps exist and are display maps
$$\begin{array}{ccc} \Delta^+ & \xrightarrow{s^+} & \Gamma^+ \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{s} & \Gamma \end{array},$$
2. display maps are closed under composition, and
3. isomorphisms and terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.

- Definition due to Taylor[5], name due to Joyal[6] (‘a clan is a collection of families’)
- Relation to semantics of dependent type theory: display maps represent **type families**.
- Observation: clans have finite products (as pullbacks over 1).

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[5] P. Taylor. “Recursive domains, indexed category theory and polymorphism”. PhD thesis. University of Cambridge, 1987, § 4.3.2.

[6] A. Joyal. “Notes on clans and tribes”. In: *arXiv preprint arXiv:1710.10238* (2017).

Examples

- Finite-product categories $\mathcal{C}$ can be viewed as clans with $\mathcal{C}_\dagger = \{\text{product projections}\}$
- Finite-limit categories $\mathcal{L}$ can be viewed as clans with $\mathcal{L}_\dagger = \mathrm{mor}(\mathcal{L})$

We call such clans **FP-clans**, and **FL-clans**, respectively.

- The syntactic category of every Cartmell-style **generalized algebraic theory** is a clan.
- Clan for categories:

$$\begin{aligned} \mathcal{K} &= \{\text{categories free on finite graphs}\}^{\mathsf{op}} \subseteq \mathbf{Cat}^{\mathsf{op}} \\ \mathcal{K}_\dagger &= \{\text{functors induced by graph inclusions}\}^{\mathsf{op}} \end{aligned}$$

  $\mathcal{K}$ can be viewed as syntactic category of a generalized algebraic theory of categories:

  - $\vdash O \quad \mathsf{sort}$
  - $x\,y : O \vdash A(x,y) \quad \mathsf{sort}$
  - $x : O \vdash \mathrm{id}(x) : A(x,x)$
  - $x\,y\,z : O\,,\ f : A(x,y)\,,\ g : A(y,z) \vdash g \circ f : A(x,z)$
  - $w\,x\,y\,z : O\,,\ e : A(w,x)\,,\ f : A(x,y)\,,\ g : A(y,z) \vdash (g \circ f) \circ e = g \circ (f \circ e) : A(w,z)$
  - $x\,y : O\,,\ f \in A(x,y) \vdash 1 \circ f = f = f \circ 1 : A(x,y)$

  Vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.

Models

Definition

A **model** of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathbf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\mathbf{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \mathbf{Set}]$ of models is l.f.p. and contains $\mathcal{T}^{\mathsf{op}}$.
- For FP-clans $(\mathcal{C}, \mathcal{C}_\dagger)$ we have $\mathbf{Mod}(\mathcal{C}, \mathcal{C}_\dagger) = \mathbf{FP}(\mathcal{C}, \mathbf{Set})$.
- For FL-clans $(\mathcal{L}, \mathcal{L}_\dagger)$ we have $\mathbf{Mod}(\mathcal{L}, \mathcal{L}_\dagger) = \mathbf{FL}(\mathcal{L}, \mathbf{Set})$.
- $\mathbf{Mod}(\mathcal{K}, \mathcal{K}_\dagger) = \mathbf{Cat}$.

$$\begin{array}{ccc} & & \mathcal{T}^{\mathsf{op}} \\ & \overset{Z}{\swarrow} & \downarrow \\ \mathbf{Mod}(\mathcal{T}) & \subseteq & [\mathcal{T}, \mathbf{Set}] \end{array}$$

Observation

The same category of models may be represented by different clans.
For example, ordinary algebraic theories can be represented by FP-clans as well as FL-clans.

The weak factorization system

- Would like duality between clans and their categories of models.
- Since the same l.f.p. category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.
- Solution: equip the models with additional structure in form of a **weak factorization system**.

Definition

Let $\mathcal{T}$ be a clan. Define w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathbf{Mod}(\mathcal{T})$ by

- $\mathcal{F} := \mathbf{RLP}(\{Z(p) \mid p \in \mathcal{T}_\dagger\})$ class of **full maps**
- $\mathcal{E} := \mathbf{LLP}(\mathcal{F})$ class of **extensions**

I.e. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by the image of $\mathcal{T}_\dagger$ under $Z : \mathcal{T}^{\mathrm{op}} \to \mathbf{Mod}(\mathcal{T})$.

- Call $A \in \mathbf{Mod}(\mathcal{T})$ a 0-**extension**, if $(0 \to A) \in \mathcal{E}$

- E.g. corepresentables $Z(\Gamma)$ are 0-extensions since terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.
- The same weak factorization system was also introduced by S. Henry in a HoTTEST talk[7], see also[8].

[7] S. Henry, *The language of a model category*, HoTTEST seminar, Jan. 2020, https://youtu.be/7_X0qbSX1fk

[8] S. Henry. "Algebraic models of homotopy types and the homotopy hypothesis". In: *arXiv preprint arXiv:1609.04622* (2016).

Full maps

- $f : A \to B$ in $\mathbf{Mod}(\mathcal{T})$ is full iff it has the RLP with respect to all $Z(p)$ for display maps $p : \Delta \twoheadrightarrow \Gamma$.

$$\begin{array}{ccc} \mathcal{T}(\Gamma, -) & \longrightarrow & A \\ {\scriptstyle Z(p)=\mathcal{T}(p,-)}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathcal{T}(\Delta, -) & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} A(\Delta) & \xrightarrow{f_\Delta} & B(\Delta) \\ {\scriptstyle A(p)}\downarrow & & \downarrow{\scriptstyle B(p)} \\ A(\Gamma) & \xrightarrow{f_\Gamma} & B(\Gamma) \end{array}$$

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering $p : \Delta \twoheadrightarrow 1$ we see that full maps are surjective and hence regular epis.

$$\begin{array}{ccc} A(\Delta) & \xrightarrow{f_\Delta} & B(\Delta) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array} \qquad \begin{array}{ccc} A(\Delta) & \xrightarrow{f_\Delta} & B(\Delta) \\ \downarrow & & \downarrow \\ A(\Delta) \times A(\Delta) & \xrightarrow{f_\Delta \times f_\Delta} & B(\Delta) \times B(\Delta) \end{array}$$

- For FL-clans, only isos are full (consider naturality square for diagonal $\Delta \to \Delta \times \Delta$)
- For FP-clans we have

$$\begin{array}{rcl} \text{full map} & = & \text{regular epimorphism} \\ 0\text{-extension} & = & \text{projective object} \end{array}$$

Duality for clans

Theorem (F)

There is a bi-equivalence of 2-categories

$$\mathbf{Clan}_{cc} \underset{\mathcal{T} \mapsto \mathbf{Mod}(\mathcal{T})}{\overset{\{\text{compact } 0\text{-extensions}\}^{op} \leftarrow\!\shortmid\, \mathfrak{X}}{\longleftrightarrow}} \mathbf{cAlg}^{op}$$

- $\mathbf{Clan}_{cc}$ is the 2-category of clans and functors preserving 1, display maps and pullbacks of display maps
- $\mathbf{cAlg}$ is the 2-category of **clan-algebraic categories**, i.e. categories $\mathcal{X}$ equipped with a WFS $(\mathcal{E}, \mathcal{F})$ of **extensions** and **full maps**, such that
  1. $\mathfrak{X}$ is locally small and cocomplete,
  2. $\mathfrak{X}$ has a small dense family of compact 0-extensions (in particular $\mathfrak{X}$ is l.f.p.),
  3. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions, and
  4. $\mathfrak{X}$ has full and effective quotients of componentwise-full equivalence relations.
- Both directions of the proof are non-trivial, details in the appendix

Models in Higher Types

Models in higher types

Let $\mathcal{S}$ be the ∞-topos of spaces/types.

Let $\mathcal{C}_{\mathsf{Mon}}$ be the finite-product theory of monoids, and let $\mathcal{L}_{\mathsf{Mon}}$ be the finite-limit theory of monoids. Then

$$\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathbf{Set}) \simeq \mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathbf{Set})$$

but $\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ and $\mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ are different:

- $\mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ is just the category of monoids
- $\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ is the ∞-category 'A_∞-algebras', i.e. homotopy-coherent monoids.

Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon was recently discussed under the name 'animation' in[9], and earlier in[10]

9 K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: *arXiv preprint arXiv:1912.10932* (2019).

10 D. Quillen. *Homotopical algebra*. Springer, 1967.

Four clans for categories

Cat admits several clan-algebraic weak factorization systems:

- $(\mathcal{E}_1, \mathcal{F}_1)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2})\}$
- $(\mathcal{E}_2, \mathcal{F}_2)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}$
- $(\mathcal{E}_3, \mathcal{F}_3)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2})\}$
- $(\mathcal{E}_4, \mathcal{F}_4)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2}), (2 \to 1)\}$

where $\mathbb{P} = (\bullet \rightrightarrows \bullet)$.

The right classes are:

$$\begin{aligned}
\mathcal{F}_1 &= \{\text{full and surjective-on-objects functors}\} \\
\mathcal{F}_2 &= \{\text{full and bijective-on-objects functors}\} \\
\mathcal{F}_3 &= \{\text{fully faithful and surjective-on-objects functors}\} \\
\mathcal{F}_4 &= \{\text{isos}\}
\end{aligned}$$

Note that $\mathcal{F}_3$ is the class of trivial fibrations for the canonical model structure on **Cat**.

Four clans for categories

These correspond to the following clans:

$$\mathcal{T}_1 = \{\text{free cats on fin. graphs}\}^{\text{op}} \qquad \mathcal{T}_1^\dagger = \{\text{graph inclusions}\}$$

$$\mathcal{T}_2 = \{\text{free cats on fin. graphs}\}^{\text{op}} \qquad \mathcal{T}_2^\dagger = \{\text{injective-on-edges maps}\}$$

$$\mathcal{T}_3 = \{\text{f.p. cats}\}^{\text{op}} \qquad \mathcal{T}_3^\dagger = \{\text{injective-on-objects functors}\}$$

$$\mathcal{T}_4 = \{\text{f.p. cats}\}^{\text{op}} \qquad \mathcal{T}_4^\dagger = \{\text{all functors}\}$$

Models in higher types:

$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_1) = \{\text{Segal spaces}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_2) = \{\text{Segal categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_3) = \{\text{pre-categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_4) = \{\text{discrete 1-categories}\}$$

Syntax

From clans to theories

- Duality between clans and clan-algebraic categories is a **theory/model duality**, where the theories themselves are of a categorical nature.

$$\mathbf{Clan}_{cc} \underset{\mathcal{T} \mapsto \mathbf{Mod}(\mathcal{T})}{\overset{\mathrm{comp}(\mathfrak{X})^{op} \leftarrow\!\shortmid \mathfrak{X}}{\longleftrightarrow}} \mathbf{cAlg}^{op}$$

- There's also a correspondence between categorical theories (clans) and syntactic theories (GATs)
  - The syntactic category of every GAT is a clan
  - Moreover, (I think that) every clan is equivalent to the syntactic category of a GAT, giving rise to an essentially surjective map as below.

$$\begin{array}{ccccc} & & \mathbf{Clan}_{cc} & \overset{\simeq}{\longleftrightarrow} & \mathbf{cAlg}^{op} \\ & & \downarrow & & \\ \{\mathsf{GATs}\} & \twoheadrightarrow & \mathbf{Clan} & & \end{array}$$

- This map can be enhanced to an equivalence by defining 1- and 2-cells between GATs to be 1- and 2-cells between the corresponding clans.

Four GATs for categories

GAT for $\mathcal{T}_1$

- $\vdash\ O$ sort
- $x\,y : O \vdash A(x,y)$ sort
- $w\,x\,y\,z : O\,,\ e : A(w,x)\,,\ f : A(x,y)\,,\ g : A(y,z) \vdash (g \circ f) \circ e = g \circ (f \circ e) : A(w,z)$
- $x\,y : O\,,\ f \in A(x,y) \vdash 1 \circ f = f = f \circ 1 : A(x,y)$
- $x : O \vdash 1 : A(x,x)$
- $x\,y\,z : O\,,\ f : A(x,y)\,,\ g : A(y,z) \vdash g \circ f : A(x,z)$

- $\mathcal{T}_2$ should have an equivalent syntactic category but more display maps, including the diagonal

$$\delta_O = (x,x) : [x : O] \to [x\,y : O].$$

- This is not a context projection, but we can make it isomorphic to one by introducing a new type over $[x\,y : O]$ and forcing it to be isomorphic to $[x : O]$

Four GATs for categories

Additional axioms for $\mathcal{T}_2$

- $x\,y : O \vdash E(x,y)$ sort
- $x : O \vdash r : E(x,x)$
- $x\,y : O\,,\, p : E(x,y) \vdash x = y$
- $x\,y : O\,,\, p\,q : E(x,y) \vdash p = q$

- In other words, we add an extensional equality type for O 'by hand'
- With this we can show the isomorphism of contexts $[x : O] \cong [x\,y : O\,,\, p : E(x,y)]$
- Similarly, add extensional equality for morphisms to get $\mathcal{T}_e$:

Additional axioms for $\mathcal{T}_3$

- $x\,y : O\,,\, f\,g : A(x,y) \vdash F(f,g)$ sort
- $x\,y : O\,,\, f : A(x,y) \vdash s : F(f,f)$
- $x\,y : O\,,\, f\,g : A(x,y)\,,\, p : F(f,g) \vdash f = g$
- $x\,y : O\,,\, f\,g : A(x,y)\,,\, p\,q : F(f,g) \vdash p = q$

- Adding both sets of axioms yields $\mathcal{T}_4$

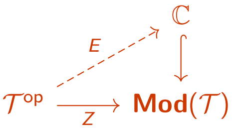
Appendix – the proof

Reconstructing the clan

Definition

Given a clan $\mathcal{T}$, let $\mathbb{C} \subseteq \mathbf{Mod}(\mathcal{T})$ be the full subcategory on **compact** 0**-extensions**.

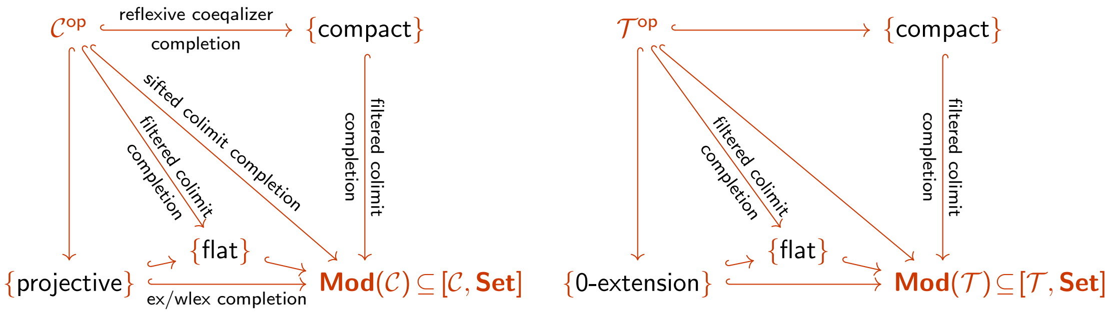
- $\mathbb{C}$ is a **coclan** with extensions as "co-display maps".
- $Z : \mathcal{T}^{\mathrm{op}} \to \mathbf{Mod}(\mathcal{T})$ factors through $\mathbb{C}$ since corepresentables $Z(\Gamma)$ are compact and 0-extensions.



- Have to show that E is a Morita equivalence, i.e. every compact 0-extension is a retract of a corepresentable.

The fat small object argument

Motivation: Subcategories of models for FP-theory $\mathcal{C}$ and clan $\mathcal{T}$.



- Flat algebras are filtered colimits of corepresentables, computed *freely* in the functor categories.
- For algebraic theories we have {projective} $\subseteq$ {flat} since
  - arbitrary free objects are filtered colimits of free objects over finite sets
  - projective objects are retracts of free objects
- In the general clan case, {0-extension} $\subseteq$ {flat} by the **fat small object argument**[11].

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[11] M. Makkai, J. Rosicky, and L. Vokrinek. "On a fat small object argument". In: *Advances in Mathematics* (2014).

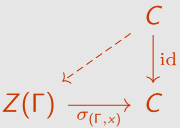
Reconstructing the clan

Theorem

The full inclusion $E : \mathcal{T}^{\mathrm{op}} \hookrightarrow \mathbb{C}$ exhibits $\mathbb{C}$ as *Cauchy-completion* of $\mathcal{T}^{\mathrm{op}}$, i.e. every compact 0-extension is a retract of a corepresentable.

Proof.

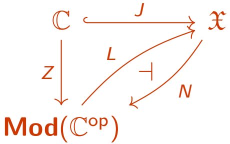
- Let $C \in \mathbb{C}$.
- Since 0-extensions are flat, $\int C$ is filtered, thus C is a filtered colimit of corepresentables.
- Since C is compact, id_C factors through a colimit inclusion map.



□

Other direction – Idea of proof

- Show that the nerve/realization adjunction



$$L(A) = \mathrm{colim}(\textstyle\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$$
$$N(X) = \mathfrak{X}(J(-), X)$$

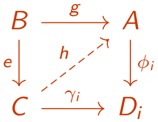
  is an equivalence.
- By density the right adjoint N is fully faithful, i.e. the counit is an isomorphism.
- It remains to show that the unit of the adjunction is an isomorphism, i.e.

$$A(C) \xrightarrow{\cong} \mathfrak{X}(C, \mathrm{colim}(\textstyle\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})).$$

  for all $A \in \mathbf{Mod}(\mathbb{C}^{\mathrm{op}})$ and $C \in \mathbb{C}$.
- The functor $\mathcal{X}(C, -)$ preserves filtered colimits and quotients of componentwise-full equivalence relations, so it suffices to decompose $\mathrm{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$ in terms of these constructions.
- This is essentially what we're doing in the following.

Jointly full cones

- Let $D : \mathcal{I} \to \mathfrak{X}$ be a diagram in an adequate category.
- A cone (A, ϕ) over D is called **jointly full**, if for every cone (C, γ), extension $e : B \to C$ and map $g : B \to A$ constituting a cone morphism $g : (B, \gamma \circ e) \to (A, \phi)$, there exists a map $h : C \to A$ such that

  commutes for all $i \in \mathcal{I}$.
- **Observation:** The cone (A, ϕ) is jointly full iff the canonical map to the limit is full.

Definition

A **nice diagram** in an adequate category $\mathfrak{X}$ is a truncated simplicial diagram

$$A_2 \;\substack{\xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2}}\; A_1 \;\substack{\xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1}}\; A_0$$

where

1. A_0, A_1, and A_2 are 0-extensions,
2. the maps $d_0, d_1 : A_1 \to A_0$ are full,
3. in the square $\begin{array}{ccc} A_2 & \xrightarrow{d_0} & A_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} \\ A_1 & \xrightarrow{d_0} & A_0 \end{array}$ the span constitutes a jointly full diagram over the cospan,
4. there exists a symmetry map $\begin{array}{ccc} A_1 & \xrightarrow{d_1} & A_0 \\ {\scriptstyle d_0}\downarrow & \searrow^{\sigma} & \uparrow{\scriptstyle d_0} \\ A_0 & \xleftarrow{d_1} & A_1 \end{array}$ making the triangles commute, and
5. there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \twoheadrightarrow A_1$ constituting a jointly full cone over the diagram

$$\begin{array}{ccccc} A_1 & & & & A_1 \\ {\scriptstyle d_0}\downarrow & \searrow^{d_1} & & \swarrow_{d_0} & \downarrow{\scriptstyle d_1} \\ A_0 & & & & A_0 \end{array}.$$

Nice diagrams

Lemma

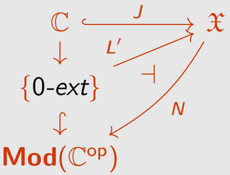
For any nice diagram, the pairing $A_1 \xrightarrow{\langle d_0, d_1 \rangle} A_0 \times A_0$ *admits a decomposition* $A_1 \twoheadrightarrow R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$ *into a full map and a monomorphism, and* $\langle r_0, r_1 \rangle$ *is a componentwise-full equivalence relation.*

Lemma

Assume $\mathfrak{X}$ *is adequate and* $F : \mathfrak{X} \to \mathbf{Set}$ *preserves finite limits and sends full maps to surjections. Then for every nice diagram,* F *preserves coequalizers of the arrows* $d_0, d_1 : A_1 \to A_0$.

Lemma

The restriction L' *of* L *in the nerve/realization adjunction*



to 0*-extensions is fully faithful and preserves full maps and nice diagrams.*

Nice diagrams

Lemma

For every object A of an adequate category $\mathfrak{X}$ there exists a nice diagram

$$A_2 \underset{d_2}{\overset{d_0}{\underset{\xleftarrow{s_1}}{\overset{\xleftarrow{s_0}}{\xrightarrow{d_1}}}}} A_1 \underset{d_1}{\overset{d_0}{\underset{\xleftarrow{s_0}}{\rightrightarrows}}} A_0$$

such that A is the coequalizer of $d_0, d_1 : A_1 \to A_0$.

Proof.

- A_0 is given by covering A by a 0-extension, i.e. factoring $0 \to A$ as $0 \hookrightarrow A_0 \overset{e}{\twoheadrightarrow} A$.
- A_1 is given by covering the kernel of $A_0 \twoheadrightarrow A$ by a 0-extension

$$\begin{array}{ccccccc} 0 & \hookrightarrow & A_1 & \twoheadrightarrow & R & \xrightarrow{r_0} & A_0 \\ & & & & {\scriptstyle r_1}\downarrow & \lrcorner & \downarrow{\scriptstyle e} \\ & & & & A_0 & \xrightarrow{e} & A \end{array}$$

- A_2 is given by covering the following pullback:

$$\begin{array}{ccccccc} 0 & \hookrightarrow & A_2 & \twoheadrightarrow & \bullet & \longrightarrow & A_1 \\ & & & & \downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ & & & & A_1 & \xrightarrow{d_1} & A_0 \end{array}$$

□

The proof

Proof of the theorem.

Let $\mathbb{C} \subseteq \mathfrak{X}$ be the co-clan of compact 0-extensions. It remains to show that

$$AC \cong \mathfrak{X}(C, LA).$$

for all $A \in \mathbf{Mod}(\mathbb{C}^{\mathrm{op}})$ and $C \in \mathbb{C}$. Let $A_\bullet$ be a nice diagram with coequalizer A. We have

$$\begin{aligned}
\mathfrak{X}(C, LA) &= \mathfrak{X}(C, L(\mathrm{coeq}(A_1 \rightrightarrows A_0))) && \text{since } A = \mathrm{coeq}(A_1 \rightrightarrows A_0) \\
&\cong \mathfrak{X}(C, \mathrm{coeq}(LA_1 \rightrightarrows LA_0)) && \text{since } L \text{ preserves colimits} \\
&\cong \mathrm{coeq}(\mathfrak{X}(C, LA_1) \rightrightarrows \mathfrak{X}(C, LA_0)) && \text{since } \mathfrak{X}(C, -) \text{ preserves coeqs of nice diags} \\
&\cong \mathrm{coeq}(A_1 C \rightrightarrows A_0 C) && \text{since } LA_i = \mathrm{colim}(\textstyle\int A_i \to \mathbb{C} \to \mathfrak{X}) \text{ filtered} \\
&\cong \mathrm{coeq}(\mathbf{Mod}(ZC, A_1) \rightrightarrows \mathbf{Mod}(ZC, A_0)) \\
&\cong \mathbf{Mod}(ZC, \mathrm{coeq}(A_1 \rightrightarrows A_0)) \\
&\cong \mathbf{Mod}(ZC, A)) \\
&\cong AC
\end{aligned}$$

Related work

- B. Ahrens, P. North, M. Shulman, and D. Tsementzis. "A higher structure identity principle". English. In: *Proceedings of the 2020 35th annual ACM/IEEE symposium on logic in computer science, LICS 2020, virtual event, July 8–11, 2020*. New York, NY: Association for Computing Machinery (ACM), 2020
- I. Di Liberti and J. Rosický. "Enriched Locally Generated Categories". In: (Sept. 2020). arXiv: 2009.10980 [math.CT]
- C.L. Subramaniam. "From dependent type theory to higher algebraic structures". In: (Oct. 2021). arXiv: 2110.02804 [math.CT]
- S. Henry. "Algebraic models of homotopy types and the homotopy hypothesis". In: *arXiv preprint arXiv:1609.04622* (2016)

Thanks for your attention!