Positive negation in constructive mathematics

Iosif Petrakis

University of Munich

COST Action CA20111
WG6 Meeting on "Syntax and Semantics of Type Theories"
Stockhol University, Department of Mathematics
20.05.2022

In constructive logic

Nelson, Markov: objected to weak intuitionistic negation

$$\neg A := A \Rightarrow \bot$$

$\neg(A \wedge B)$ does not imply $\neg A \vee \neg B$

$\neg\big(\forall_x \phi(x)\big)$ does not imply $\exists_x \neg\phi(x)$

Constructive logic with strong negation ($\neg$) was formulated.

In constructive mathematics

Brouwer, Bishop: use the weak negation, but in many cases developed a positive approach to negatively defined concepts:

denial inequality – positive inequality, apartness relation

complement of a subset – strong complement of a subset

disjoint subsets – complemented subsets

non-empty set – inhabited set

Abstract inequalities rely on $\neg$ (Bishop and Bridges 1985)

$\mathtt{f}$-inequalities are completely positively defined (Bishop 1967).

Shulman: Affine logic for CM, 2021

He showed that numerous concepts of CM arise automatically from an "antithesis" translation of affine logic into intuitionisitic logic (IL) via a Chu/Dialectica construction.

What we do is similar, but

we work within BISH, we define a strong negation and we use Rasiowa's strong implication.

Why using a strong, positive $\vee, \exists$ together with a weak and negative $\neg$ only?

Strong: $\vee, \exists, \neg, \Rightarrow$

Weak: $\vee, \exists, \neg, \Rightarrow$

Common: $\wedge, \forall$.

Formulas in BISH

Prime formulas:

$s =_{\mathbb{N}} t$, $s \neq_{\mathbb{N}} t$, where s, t are elements of $\mathbb{N}$.

Complex formulas:

If A, B are formulas, then $A \vee B, A \wedge B, A \Rightarrow B$ are formulas.

If S is a set and $\phi(x)$ is a formula, for every variable x of set S, then $\exists_{x \in S}\big(\phi(x)\big)$ and $\forall_{x \in S}\big(\phi(x)\big)$ are formulas.

Weak negation in BISH

$$\neg A := A \Rightarrow \bot,$$

$$\bot := 0 =_{\mathbb{N}} 1$$

$$\top := 0 \neq_{\mathbb{N}} 1$$

Strong negation in BISH

$$\neg(s =_{\mathbb{N}} t) := s \neq_{\mathbb{N}} t \;\; \& \;\; \neg(s \neq_{\mathbb{N}} t) := s =_{\mathbb{N}} t.$$

$$\neg(A \vee B) := \neg A \wedge \neg B$$

$$\neg(A \wedge B) := \neg A \vee \neg B$$

$$\neg(A \Rightarrow B) := A \wedge \neg B$$

$$\neg\Big(\exists_{x \in S} \phi(x)\Big) := \forall_{x \in S}\big(\neg\phi(x)\big)$$

$$\neg\Big(\forall_{x \in S} \phi(x)\Big) := \exists_{x \in S}\big(\neg\phi(x)\big)$$

Proposition

Let A be a formula of BISH*.*

(i) $\neg\neg A \Rightarrow A$.

(ii) $\neg A \Rightarrow \neg A$.

(iii) $A \wedge \neg A \Rightarrow \bot$.

$$\neg\neg A := \neg(A \Rightarrow 0 =_{\mathbb{N}} 1) := A \wedge 0 \neq_{\mathbb{N}} 1 \Leftrightarrow A$$

The strong inequality of a defined set $(X, =_X)$ is defined by

$$x \neq_X y := \neg(x =_X y).$$

We call $(X, =_X, \neq_X)$ a set with inequality.

We call $(X, =_X, \neq_X)$ a strong set.

$$\neg(x \neq_X y) \Rightarrow x =_X y$$

Richman: to define an inequality for every set would be "cumbersome and easily forgotten".

In most cases, but not all, the sets with inequality considered are strong!

The strong inequality of $\mathbb{R}$

$$x =_{\mathbb{R}} y :\Leftrightarrow \forall_{n \in \mathbb{N}^+} \left(|x_n - y_n| \leqslant \frac{2}{n} \right)$$

$$\begin{aligned} x \neq_{\mathbb{R}} y &:\Leftrightarrow \exists_{n \in \mathbb{N}^+} \left(\neg \left(|x_n - y_n| \leqslant \frac{2}{n} \right) \right) \\ &\Leftrightarrow \exists_{n \in \mathbb{N}^+} \left(|x_n - y_n| > \frac{2}{n} \right) \\ &\Leftrightarrow: |x - y| > 0 \\ &\Leftrightarrow: x \neq_{\mathbb{R}} y \end{aligned}$$

The strong inequality of the product set

$$
\begin{aligned}
(x, y) \neq_{X\times Y} (x', y') &:= \neg\big[(x, y) =_{X\times Y} (x', y')\big] \\
&:= \neg\big[x =_X x' \wedge y =_Y y'\big] \\
&:= x \neq_X x' \vee y \neq_Y y'.
\end{aligned}
$$

If $\neq_X$ and $\neq_Y$ are extensional, then $\neq_{X\times Y}$ is extensional.

The strong inequality of the function set

$$
\begin{aligned}
f \neq_{\mathbb{F}(X,Y)} g &:= \neg\big[f =_{\mathbb{F}(X,Y)} g\big] \\
&:= \neg\Big[\forall_{x \in X}\big(f(x) =_Y g(x)\big)\Big] \\
&:= \exists_{x \in X} \neg\big[f(x) =_Y g(x)\big] \\
&:= \exists_{x \in X}\big[f(x) \neq_Y g(x)\big].
\end{aligned}
$$

If $\neq_Y$ is extensional, then $\neq_{\mathbb{F}(X,Y)}$ is extensional.

Strong inequality need not be an apartness relation

$$
\begin{aligned}
(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) &:\Leftrightarrow i =_I j \wedge \lambda_{ij}(x) =_{\lambda_0(j)} y \\
&:\Leftrightarrow i =_I j \wedge [i =_I j \Rightarrow \lambda_{ij}(x) =_{\lambda_0(j)} y]
\end{aligned}
$$

$$
(i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i \neq_I j \vee \big(i =_I j \wedge \lambda_{ij}(x) \neq_{\lambda_0(j)} y\big)
$$

Even if $\neq_I$ and $\neq_{\lambda_0(j)}$ are apartness relations, I needs to be discrete to get an apartness on the Sigma-set of the family of sets over I.

Rasiowa's strong implication in BISH

$$A \Rightarrow B := (A \Rightarrow B) \wedge (\neg B \Rightarrow \neg A)$$

Functions between sets

An a.r. $f\colon (X, =_X) \to (Y, =_Y)$ is a function, if

$$x =_X x' \Rightarrow f(x) =_Y f(x').$$

A function $f\colon (X, =_X, \neq_X) \to (Y, =_Y, \neq_Y)$ is strongly extensional, if

$$f(x) \neq_Y f(x') \Rightarrow x \neq_X x'.$$

A function $f\colon (X, =_X, \neq_X) \to (Y, =_Y, \neq_Y)$ is strong, if

$$f(x) \neq_Y f(x') \Rightarrow x \neq_X x'.$$

Hence an a.r. $f\colon (X, =_X, \neq_X) \to (Y, =_Y, \neq_Y)$ is a strong function, if

$$x =_X x' \Rightarrow f(x) =_Y f(x').$$

In BISH we cannot accept that all functions are strong

The following are equivalent:

(i) Markov's principle.

(ii) Every function $f : (\mathbb{R}, =_{\mathbb{R}}, \neq_{\mathbb{R}}) \to (\mathbb{R}, =_{\mathbb{R}}, \neq_{\mathbb{R}})$ is strong.

(iii) $\neg(x =_{\mathbb{R}} 0) \Rightarrow \neg(x =_{\mathbb{R}} 0)$.

(iv) $\neg(x \leqslant y) \Rightarrow x > y$.

Hence, in BISH we cannot accept:

$$(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A),$$

$$\neg A \Rightarrow \neg A,$$

$$(\neg\neg A) \Rightarrow A,$$

as $\neg(x > y) \Leftrightarrow x \leqslant y$.

You need IL to show that a constant function is strong!

The category of strong sets and strong functions

Set : category of sets and functions

$\mathbf{Set}^{\mathrm{se}}$: category of sets with inequality and s.e. functions

Set : category of strong sets and strong functions

Constructive measure theory within $\mathbf{Set}^{\mathrm{se}}$

The strong complement of a subset

If $(X, =_X, \neq_X)$ is a strong set, and $(A, i_A) \subseteq X$, then for every $x \in A$ let the pseudo-membership

$$x \in A \equiv \exists_{a \in A}\big(i_A(a) =_X x\big).$$

hence,

$$\begin{aligned} x \notin A &:= \neg\big[x \in A\big] \\ &:= \neg\big[\exists_{a \in A}\big(i_A(a) =_X x\big)\big] \\ &:= \forall_{a \in A}\neg\big(i_A(a) =_X x\big) \\ &:= \forall_{a \in A}\big(i_A(a) \neq_X x\big). \end{aligned}$$

If $\neq_X$ is extensional, the strong complement of A is defined by

$$A^{\neq} := \{x \in X \mid x \notin A\}.$$

The proof of $\sqrt{2} \notin \mathbb{Q}$ more informative than the proof of $\sqrt{2} \notin \mathbb{Q}$.

$P(x)$ extensional property on $(X, =_X, \neq_X)$.

$$X_P := \{x \in X \mid P(x)\}$$

$$x \in X_P := P(x)$$

$$\begin{aligned} X_P \text{ is empty } &:= \neg \exists_{x \in X} (x \in X_P) \\ &:= \neg \exists_{x \in X} P(x) \\ &:= \forall_{x \in X} \neg P(x). \end{aligned}$$

Complemented subsets

If $(A, i_A^X), (B, i_B^X) \subseteq (X, =_X, \neq_X)$, then

$$A \cap B := \{(a, b) \in A \times B \mid i_A^X(a) =_X i_B^X(b)\}.$$

$$\begin{aligned}
A \cap B \text{ is empty } &:= \forall_{(a,b)\in A\times B}\big(\neg i_A^X(a) =_X i_B^X(b)\big) \\
&:= \forall_{(a,b)\in A\times B}\big(i_A^X(a) \neq_X i_B^X(b)\big) \\
&\Leftrightarrow \forall_{a\in A}\forall_{b\in B}\big(i_A^X(a) \neq_X i_B^X(b)\big).
\end{aligned}$$

Strong uniqueness

If $(X, =_X, \neq_X)$, let

$$\exists!_{x \in X} \phi(x) := \phi(x_0) \wedge \forall_{x \in X} \big(x \neq_X x_0 \Rightarrow \neg\phi(x)\big)$$

Tight concepts

$\neg A$ is tight if and only if

$$(\neg\neg A) \Rightarrow A$$

If $C \subseteq X$, then $C^{\neq}$ is tight, if

$$\left(C^{\neq}\right)' \subseteq C$$

If C is a closed and located subset of a metric space, then $C^{\neq}$ is tight (without CC).

Conclusions/Questions

- $\neg$ is a heuristic method of defining positively various concepts of CM.
- The use of $\neg$ forces us to do better and more informative proofs.
- It permits the distinction between strong, weak and tight concepts.
- CM is mathematics with IL. More distinctions need to be kept: no choice, predicativity, positivity.
- Is an intuitionistic proof using $\neg$ fully constructive (Griss)?
- MLTT and HoTT claim that they can serve as a foundation for all mathematics (constructive and classical). As there is no canonical inequality associated to the equality type $a =_A b$, can intensional MLTT (and HoTT) capture strong negation?

S. Akama: Constructive Predicate Logic with Strong Negation and Model Theory, Notre Dame Journal of Formal Logic 29, 1988, 18–27.

E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

E. Bishop, D. S. Bridges: *Constructive Analysis*, Grundlehren der Math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.

G. F. Griss: Negationless intuitionistic mathematics I, Indagationes Mathematicae 8, 1946, 675-681.

A. A. Markov: Konstruktivnaja logika, Usp. Mat. Nauk 5, 1950, 187–188.

R. Mines, F. Richman, W. Ruitenburg: *A course in constructive algebra*, Springer Science+Business Media New York, 1988.

I. Petrakis: *Families of Sets in Bishop Set Theory*, Habilitationsschrift, LMU, Munich, 2020, available at `https://www.mathematik.uni-muenchen.de/~petrakis/`.

I. Petrakis: Direct spectra of Bishop spaces and their limits, Logical Methods in Computer Science, Volume 17, Issue 2, 2021, pp. 4:1-4:50.

I. Petrakis: Proof-relevance in Bishop-style constructive mathematics, Mathematical Structures in Computer Science, 2022, to appear.

I. Petrakis, D. Wessel: Algebras of complemented subsets, CiE 2022, to appear.

H. Rasiowa: *An Algebraic Approach to Non-Classical Logics*, North-Holland, 1974.

M. Shulman: Affine Logic for Constructive Mathematics, arXiv:1805.07518v2, 2021.