Higher geometric sheaf theories

Towards geometric Homotopy Type Theories?

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The theory of classifying 1-toposes

The theory of classifying toposes is a translation of logical structure on three levels:

Fragments of 1st order logic (Symbolic syntax)	Categorical syntax	Classifying toposes
Cartesian theories	Left exact categories	Presheaf toposes over lex categories
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The classifying topos associated to a [label] theory T with syntactic site $\mathcal{C}(T)$ is a topos of sheaves

$$\mathrm{Sh}(\mathcal{C}(T)) \hookrightarrow \hat{\mathcal{C}}$$

on $\mathcal{C}(T)$ such that

- the Yoneda embedding factors through $\mathrm{Sh}(\mathcal{C}(T))$ (so its associated Grothendieck topology is always sub-canonical), and
- for every other topos $\mathcal{E}$, restriction along this Yoneda-embedding induces an equivalence

$$\mathrm{LTop}(\mathrm{Sh}(\mathcal{C}(T)), \mathcal{E}) \simeq \text{[label]-}\mathrm{Cat}(\mathcal{C}(T), \mathcal{E}).$$

Thus, all these fragments of 1st order theories have in common that their categorical semantics is preserved by push-foward along geometric morphisms between toposes.

Many theories in the wild fall into one of these classes of 1st order theories, e.g. fields (with finite characteristic), (local) rings, torsion(free) abelian groups, …

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By construction, for a theory T, its classifying topos is the initial topos equipped with a T-model. E.g. it is the initial topos with a (local) ring object, … Thus,

- the theory of classifying toposes allows to synthesize fairly involved internal categorical objects in toposes by way of a fairly easy-to-manage syntax;
- knowing that a given topos is the classifying topos of some sort of theory can be valuable information about the topos itself from a practical perspective as well. E.g. obviously, if it classifies a cartesian theory, it is a presheaf topos and hence as tame as it gets. Or, if it classifies a coherent theory, it still always has enough points (viz. Deligne Completeness Theorem).

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An analogous translation in the basic cartesian case has implicitly been established in the literature: By Kapulkin, Lumsdaine ([5]) and Kapulkin, Szumiło ([4]) on the one hand, and by standard ∞-categorical constructions on the other.

Fragments of intensional type theory (Symbolic syntax)	∞-Categorical syntax	∞-Topos theoretic models
$\mathrm{MLTT}^{\Sigma,\mathrm{Id},1}$	Left exact ∞-categories	Presheaf ∞-toposes over lex ∞-categories
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Here, the same realm of potential applications applies:

1. Higher geometric theories for structures arising in for instance higher algebra?
2. Again, knowing that an ∞-topos is the classifying ∞-topos of some sort of theory yields intimate information about the topos itself. For instance,

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Higher geometric categories and their sheaf theories

What is a higher geometric ∞-category?

Approach: Replace the categorical interpretation of symbolic predicates as sub-objects by the proof-relevant interpretation as general arrows, and require suitable categorical structure not for the sub-object posets only but for the full slices.

This underlies for example Anel and Joyal's definition of ∞-logoi as ∞-categorical pretoposes ([1]).

Definition

Given a regular cardinal κ, an ∞-category $\mathcal{C}$ is κ-*geometric* if it is left exact and has universal κ-small colimits.

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Idea: In all examples of classifying 1-toposes, the according syntactic sites are sites of colimit covers: Every notion of theory determines some idiosyncratic shape of diagram that we define to be covering over their colimit.

The consideration of more general shapes of diagrams than those considered in the ordinary categorical setting is due to proof relevance of the syntax, which is reflected by the fact that ∞-toposes are generally not topological over their canonical base.

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Higher covering diagrams

Naive idea: Take all κ-small diagrams.

Problem: This generally does not yield a sheaf theory, in the sense that its ∞-category of sheaves is not always an ∞-topos: Whenever $\mathcal{C}$ is locally presentable, it is as far away from being an ∞-topos as $\mathcal{C}$ itself.

Definition

A (κ-)small *higher covering diagram* in an ∞-category $\mathcal{C}$ with pullbacks and (κ-)small colimits is a diagram $F\colon I \to \mathcal{C}$ such that

- The ∞-category I has pullbacks and F preserves them;
- F covers not only its colimit, but it "locally covers" all iterated pullbacks of components F_i, F_j over $\mathrm{colim} F$ as well.

Say a presheaf $F\colon \mathcal{C}^{op} \to \mathcal{S}$ is a *higher (κ-)geometric sheaf* if takes colimits of (κ-)small higher covering diagrams in $\mathcal{C}$ to limits of spaces.

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Let $\mathrm{Sh}(\mathcal{C}) \subseteq \hat{\mathcal{C}}$ denote the ∞-category of such higher (κ-)geometric sheaves.

Theorem

Whenever $\mathcal{C}$ is a small κ-geometric ∞-category, the ∞-category $\mathrm{Sh}(\mathcal{C})$ is a left exact localization of $\hat{\mathcal{C}}$ and hence an ∞-topos.

Proposition

The ∞-topos $\mathrm{Sh}(\mathcal{C})$ over a κ-geometric ∞-category $\mathcal{C}$ is generally not hypercomplete (and so does generally not have enough points).

Proposition

The topological part of $\mathrm{Sh}(\mathcal{C})$ over any κ-geometric ∞-category $\mathcal{C}$ is given exactly by the ordinary geometric Grothendieck topology on $\mathcal{C}$ (whenever κ is uncountable!). The associated cotopological localization is generally non-trivial.

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Thus, $\mathrm{Sh}(\mathcal{C})$ is *canonical* for ∞-toposes $\mathcal{C}$. Non-triviality of the cotopological localization above implies that the ordinary geometric Grothendieck topology is not!

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Corollary

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$$U \colon \mathrm{LTop} \to \mathrm{GeoCat}_{(\kappa)}$$

has a left adjoint given objectwise by $\mathrm{Sh}(\mathcal{C})$. *For* κ *proper class sized, the forgetful functor* U *is fully faithful.*

Question: Is a corresponding geometric $\mathrm{MLTT}^{\Sigma,\mathrm{Id},1,\mathrm{colim}}$ feasible, where the higher inductive type former "colim" is defined for all inputs of (finite) higher covering diagrams?

Thank you!

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