Normalization for initial space-valued models of type theories

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WG6 kick-off meeting

Coherence problem

Fix a type theory $\mathcal{T}$.

Construction

- $\mathbf{I}(\mathcal{T})$ the initial set-valued model of $\mathcal{T}$
- $\mathbf{I}_\infty(\mathcal{T})$ the initial space-valued model of $\mathcal{T}$

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- $\mathbf{I}(\mathcal{T})$ the initial set-valued model of $\mathcal{T}$
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Question (Coherence problem)

$\mathbf{I}_\infty(\mathcal{T}) \simeq \mathbf{I}(\mathcal{T})$? *Equivalently, is* $\mathbf{I}_\infty(\mathcal{T})$ *set-valued?*

Then $\mathbf{I}(\mathcal{T}) \simeq \mathbf{I}_\infty(\mathcal{T}) \to \mathcal{M}$ for an arbitrary space-valued model $\mathcal{M}$.

Solution to the coherence problem

Want to calculate path spaces of $\mathbf{I}_\infty(\mathcal{T})$ and see the truncation levels of them.

Problem

$\mathbf{I}_\infty(\mathcal{T})$ *is to be a higher inductive type (Altenkirch and Kaposi 2016), so direct calculation of its path spaces is hard.*

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Idea (Higher normalization)

Show that every type or term in $\mathbf{I}_\infty(\mathcal{T})$ has a unique normal form.

- Path spaces of $\mathbf{I}_\infty(\mathcal{T})$ become equivalent to ones between normal forms.
- The space of normal forms is an (non-higher) inductive type.
- Calculation of path spaces of inductive types is straightforward.
- Cf. Decidability of judgmental equality by normalization.

How does normalization work?

Some recent developments in normalization (and more)

- Relative induction principles of Bocquet, Kaposi, and Sattler (2021): a universal property of the *category of renamings*.
- Synthetic Tait computability of Sterling (2021) and his collaborators: type theory for constructing *logical predicates*.

Observation

These are suitable for higher-dimensional analogue/generalization.

On construction of higher objects

We often have to construct an object with infinite tower of coherent homotopies. To avoid coherence issues, either

1. spell out a universal property and apply the adjoint functor theorem; or
2. use the internal language of some ∞-topos.

Theorem (Shulman 2019)

Any ∞-topos admits an interpretation of type theory with univalent universes and a lot of type constructors.

Outline of normalization proof

1. The initial space-valued model $\mathbf{I}_\infty(\mathcal{T})$ is given.
2. Go to an ∞-topos $\mathbf{X}$ where we define normal forms inductively.
3. Do something in $\mathbf{X}$.
4. Going back outside, we get a *normalization model* $\mathbf{N}_\infty(\mathcal{T})$ and then a morphism $\mathbf{I}_\infty(\mathcal{T}) \to \mathbf{N}_\infty(\mathcal{T})$ by initiality. This shows the *existence* of normal forms.
5. Go to another ∞-topos $\mathbf{Y} \supset \mathbf{X}$ to prove the *uniqueness* of normal forms.
6. Go back to $\mathbf{X}$ and show the type of normal forms is 0-truncated.
7. Going back outside, we get the coherence theorem.

∞-CwFs

Definition

An *∞-category with families (∞-CwF)* $\mathcal{M}$ consists of:

- an ∞-category $\mathbf{Ctx}_{\mathcal{M}}$ with a terminal object;
- a map $\mathrm{p}_{\mathcal{M}} : \mathrm{Tm}_{\mathcal{M}} \to \mathrm{Ty}_{\mathcal{M}}$ of (space-valued) presheaves over $\mathbf{Ctx}_{\mathcal{M}}$

(such that $\mathrm{p}_{\mathcal{M}}$ is representable).

Definition

A *space-valued model of* $\mathcal{T}$ is an ∞-CwF $\mathcal{M}$ equipped with some maps of presheaves over $\mathbf{Ctx}_{\mathcal{M}}$ and homotopies between them to model type-theoretic operators.

∞-CwFs

- This definition is an ∞-version of natural models (Awodey 2018; Fiore 2012), which are equivalent to CwFs.
- The theory of CwFs is generalized/essentially algebraic, as presented originally by Dybjer (1996).
- The "∞-theory" of ∞-CwFs is to be generalized/essentially algebraic. In particular:

Fact

The initial space-valued model $\mathbf{I}_\infty(\mathcal{T})$ of $\mathcal{T}$ exists.

CwFs

Definition

A 1-*category with families (*1-*CwF)* is an ∞-CwF $\mathcal{M}$ such that $\mathbf{Ctx}_{\mathcal{M}}$ is a 1-category and $\mathrm{Ty}_{\mathcal{M}}$ and $\mathrm{Tm}_{\mathcal{M}}$ are set-valued presheaves.

Definition

A *set-valued model of* $\mathcal{T}$ is a space-valued model of $\mathcal{T}$ whose underlying ∞-CwF is a 1-CwF.

Fact

The initial set-valued model $\mathbf{I}(\mathcal{T})$ *of* $\mathcal{T}$ *exists. By definition, we have a unique morphism* $\mathbf{I}_{\infty}(\mathcal{T}) \to \mathbf{I}(\mathcal{T})$.

Coherence problem

Proposition

The following are equivalent.

1. $\mathbf{I}_\infty(\mathcal{T}) \to \mathbf{I}(\mathcal{T})$ *is an equivalence.*
2. $\mathbf{I}_\infty(\mathcal{T})$ *is set-valued.*
3. *The presheaves* $\mathrm{Ty}_{\mathbf{I}_\infty(\mathcal{T})}$ *and* $\mathrm{Tm}_{\mathbf{I}_\infty(\mathcal{T})}$ *are set-valued.*

Slightly simplified.

Question (Coherence problem)

Are $\mathrm{Ty}_{\mathbf{I}_\infty(\mathcal{T})}$ *and* $\mathrm{Tm}_{\mathbf{I}_\infty(\mathcal{T})}$ *0-truncated in the* ∞*-topos* $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty}(\mathcal{T}))$ *of presheaves over* $\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})}$*?*

Internal models

Inside an ∞-topos $\mathcal{X} \supset \mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})})$, $\mathbf{I}_\infty(\mathcal{T})$ looks like a *logical framework encoding* (Harper, Honsell, and Plotkin 1993; Nordström, Petersson, and Smith 1990).

$$\begin{aligned} \mathtt{Ty} &: \mathcal{U} \\ \mathtt{Tm} &: \mathtt{Ty} \to \mathcal{U} \\ &\vdots \end{aligned}$$

This *can be axiomatized in type theory*. Let us call such a structure an *internal model of $\mathcal{T}$ in $\mathcal{X}$*.

Externalizing internal models

Conversely, given an internal model $(\mathtt{Ty}, \mathtt{Tm}, \dots)$ in $\mathcal{X}$, we have a space-valued model $\mathcal{M}$ by Yoneda.

$$
\begin{aligned}
\mathbf{Ctx}_{\mathcal{M}} &\subset \mathcal{X} \\
\mathrm{Ty}_{\mathcal{M}}(\Gamma) &= \mathrm{Map}_{\mathcal{X}}(\Gamma, \mathtt{Ty}) \\
\mathrm{Tm}_{\mathcal{M}}(\Gamma) &= \mathrm{Map}_{\mathcal{X}}(\Gamma, \textstyle\sum_{A:\mathtt{Ty}} \mathtt{Tm}(A)) \\
&\vdots
\end{aligned}
$$

Cf. Voevodsky's universe method (Voevodsky 2015). (We can choose for $\mathbf{Ctx}_{\mathcal{M}}$ an arbitrary full subcategory of $\mathcal{X}$ closed under context comprehension.)

Constructing space-valued models

Useful construction of space-valued models.

1. Regard $\mathbf{I}_\infty(\mathcal{T})$ as an internal model in $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})})$.
2. Embed $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})})$ into another ∞-topos $\mathcal{X}$ if necessary.
3. Do something in $\mathcal{X}$ to get an internal model in $\mathcal{X}$.
4. Externalize the internal model.

Non-stability of normal forms

Where should normal forms live?

Observation

Normal forms are NOT stable under substitution, so they cannot live in $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}(\mathcal{T})})$.

Example

$f a$ is in normal form when f is a variable and a is in normal form, but $(f a)[f :\equiv \lambda x.b] \equiv (\lambda x.b)a$ is not.

Category of renamings

Observation

Normal forms are, however, stable under renaming of variables, so they live in another presheaf topos $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{R}_{\mathrm{syn}}(\mathcal{T})})$.

$\mathbf{R}_{\mathrm{syn}}(\mathcal{T})$ is a CwF of *renamings* and syntactically defined.

- Objects of $\mathbf{Ctx}_{\mathbf{R}_{\mathrm{syn}}(\mathcal{T})}$ are the same as $\mathbf{Ctx}_{\mathbf{I}(\mathcal{T})}$, but morphisms are only renamings of variables.
- $\mathrm{Ty}_{\mathbf{R}_{\mathrm{syn}}(\mathcal{T})}(\Gamma) = \mathrm{Ty}_{\mathbf{I}(\mathcal{T})}(\Gamma)$.
- $\mathrm{Tm}_{\mathbf{R}_{\mathrm{syn}}(\mathcal{T})}(\Gamma)$ is the set of variables in Γ.

(There is also an inductive definition (Altenkirch and Kaposi 2017).)

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Problem

The syntactic construction is not suitable for ∞-analogue.

Category of renamings, categorically

Definition (Bocquet, Kaposi, and Sattler 2021)

We define $\mathbf{R}(\mathcal{T})$ to be the initial CwF equipped with a morphism $\varepsilon : \mathbf{R}(\mathcal{T}) \to \mathbf{I}(\mathcal{T})$ such that $\mathrm{Ty}_{\mathbf{R}(\mathcal{T})}(\Gamma) \cong \mathrm{Ty}_{\mathbf{I}(\mathcal{T})}(\varepsilon(\Gamma))$.

- Intuitively, terms of $\mathbf{R}(\mathcal{T})$ are variables because they are only constructed by structural rules.
- We actually do not care whether $\mathbf{R}_{\mathrm{syn}}(\mathcal{T}) \simeq \mathbf{R}(\mathcal{T})$. The latter exists by the adjoint functor theorem, and all we need in the normalization proof follow from the universal property.

∞-category of renamings, ∞-categorically

Definition

We define $\mathbf{R}_\infty(\mathcal{T})$ to be the initial ∞-CwF equipped with a morphism $\varepsilon : \mathbf{R}_\infty(\mathcal{T}) \to \mathbf{I}_\infty(\mathcal{T})$ such that $\mathrm{Ty}_{\mathbf{R}_\infty(\mathcal{T})}(\Gamma) \simeq \mathrm{Ty}_{\mathbf{I}_\infty(\mathcal{T})}(\varepsilon(\Gamma))$.

Fact

$\mathbf{R}_\infty(\mathcal{T})$ *exists.*

Relative induction principle

Remark

The universal properties of $\mathbf{I}_\infty(\mathcal{T})$ and $\mathbf{R}_\infty(\mathcal{T})$ are packed into a *relative induction principle* (Bocquet, Kaposi, and Sattler 2021).

Type theory for multiple ∞-topoi?

We now have two ∞-topoi:

- $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})})$ where $\mathbf{I}_\infty(\mathcal{T})$ is internalized;
- $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{R}_\infty(\mathcal{T})})$ where the type of normal forms is to be defined.

The morphism $\varepsilon : \mathbf{R}_\infty(\mathcal{T}) \to \mathbf{I}_\infty(\mathcal{T})$ induces the base change

$$\varepsilon^* : \mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})}) \to \mathbf{Psh}(\mathbf{Ctx}_{\mathbf{R}_\infty(\mathcal{T})}).$$

The construction of a normalization model will use objects from both sides.

Problem

What is an internal language for multiple ∞-topoi related to each other?

Artin gluing

Fact (cf. SGA4, Elephant A4.5)

Let $F^* : \mathcal{X} \to \mathcal{Y}$ *be a functor between* ∞*-topoi preserving finite limits and small colimits.*

1. *The Artin gluing* $\mathbf{Gl}(F^*)$ *is an* ∞*-topos.*
2. $\mathbf{Gl}(F^*)$ *has a special subterminal object* $P \in \mathbf{Gl}(F^*)$*.*
3. $\mathcal{X} \xrightarrow{\simeq} \mathbf{Gl}(F^*)_{/P} \underset{\perp}{\overset{\longleftarrow}{\hookrightarrow}} \mathbf{Gl}(F^*)$ *(open subtopos)*
4. $\mathcal{Y} \xrightarrow{\simeq} \{A \in \mathbf{Gl}(F^*) \mid A^P \simeq 1\} \underset{\perp}{\overset{\longleftarrow}{\hookrightarrow}} \mathbf{Gl}(F^*)$ *(closed subtopos)*
5. *The composite* $\mathcal{X} \hookrightarrow \mathbf{Gl}(F^*) \to \mathcal{Y}$ *is equivalent to* F^**.*

F^* is reconstructed from the subterminal $P \in \mathbf{Gl}(F^*)$.

Artin gluing, internally

In univalent type theory, let P be a proposition. Define subuniverses.

$$\mathcal{U}_{\mathfrak{O}} \equiv \{A : \mathcal{U} \mid \lambda x.\lambda\_.x : A \to (P \to A) \text{ is an equivalence}\} \qquad (\textit{open} \text{ subuniverse})$$

$$\mathcal{U}_{\mathfrak{C}} \equiv \{A : \mathcal{U} \mid (P \to A) \text{ is contractible}\} \qquad (\textit{closed} \text{ subuniverse})$$

They have reflectors $\mathfrak{O}(A) \equiv (P \to A)$ and $\mathfrak{C}(A) \equiv (A +_{A \times P} P)$.

Observation

$\mathbf{Gl}(F^*)$ *internally sees the diagram* $\mathcal{X} \xrightarrow{F^*} \mathcal{Y}$ *through the internal diagram* $\mathcal{U}_{\mathfrak{O}} \hookrightarrow \mathcal{U} \xrightarrow{\mathfrak{C}} \mathcal{U}_{\mathfrak{C}}$.

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Idea (Higher synthetic Tait computability)

Use univalent type theory $+ (P : \mathtt{Prop})$ *as an internal language of* $\mathbf{Gl}(F^*)$.

STC vs higher STC

Some difference from Sterling's synthetic Tait computability.

- Sterling uses *extensional* type theory for glued 1-topoi, while we use *intensional* type theory for glued ∞-topoi.
- Strict equality xor univalence.

Example

- Univalence implies $(\mathcal{U}_i)_{\mathfrak{C}} \in (\mathcal{U}_{i+1})_{\mathfrak{C}}$.
- In extensional type theory, we can still find a closed universe of closed types, using *realignment*.

First working ∞-topos

Recall

$$\varepsilon^* : \mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})}) \to \mathbf{Psh}(\mathbf{Ctx}_{\mathbf{R}_\infty(\mathcal{T})}).$$

Our first working ∞-topos is $\mathbf{X} := \mathbf{Gl}(\varepsilon^*)$.

Axiom

1. $\mathrm{P} : \mathtt{Prop}$
2. $\mathtt{Ty} : \mathcal{U}_{\mathfrak{O}}$
3. $\mathtt{Tm} : \mathtt{Ty} \to \mathcal{U}_{\mathfrak{O}}$
4. $\mathtt{IsVar} : \prod_{A:\mathtt{Ty}} \mathtt{Tm}(A) \to \mathcal{U}_{\mathfrak{C}}$

$\vdots$

Normalization model, internally

TODO

Construct in $\mathbf{X}$

- *three mutually inductive types*
  - $\mathtt{IsNfTy}(A)$ *(a type* A *is in normal form)*
  - $\mathtt{IsNfTm}(a)$ *(a term* a *is in normal form)*
  - $\mathtt{IsNeTm}(a)$ *(a term* a *is neutral)*

  in $\mathcal{U}_{\mathfrak{C}}$*;*
- *an internal normalization model*

following e.g. Gratzer (2021) and Sterling and Angiuli (2021).

Normalization model

We have an externalization $\mathbf{N}_\infty(\mathcal{T})$ of the internal normalization model. By initiality,

$$\begin{array}{ccc} & \mathbf{N}_\infty(\mathcal{T}) & \\ {\overset{s}{\nearrow}} & & {\overset{\pi}{\searrow}} \\ \mathbf{I}_\infty(\mathcal{T}) & = & \mathbf{I}_\infty(\mathcal{T}). \end{array}$$

(The relative induction principle gives us some additional structure).

Second working ∞-topos

$\mathbf{N}_\infty(\mathcal{T})$ and S have enough structure to compute normal forms of types and terms. For the *uniqueness* of normal forms, we will use induction on normal forms and neutral terms in another ∞-topos $\mathbf{Y} \supset \mathbf{X}$.

- $\mathbf{N}_\infty(\mathcal{T})$ and S are NOT internalized to $\mathbf{X}$, so we need a proper extension $\mathbf{X} \subset \mathbf{Y}$.
- The construction of $\mathbf{Y}$ depends on $\mathbf{N}_\infty(\mathcal{T})$ and S, so we cannot work in $\mathbf{Y}$ from the beginning.
- (If the notion of a morphism of ∞-CwFs could be internalized, then we could stay in $\mathbf{X}$.)

(The construction of $\mathbf{Y}$ is in the appendix. We use *oplax limits*.)

Uniqueness of normal forms

Using the section S, we have *normalization maps*

$$\mathtt{normalize}_{\mathtt{Ty}} : \prod_{A:\mathtt{Ty}} \mathtt{IsNfTy}(A)$$

$$\mathtt{normalize}_{\mathtt{Tm}} : \prod_{A:\mathtt{Ty}} \prod_{a:\mathtt{Tm}(A)} \mathtt{IsNfTm}(a).$$

TODO

Show

$$\prod_{A:\mathtt{Ty}} \prod_{A^{\mathrm{nfty}}:\mathtt{IsNfTy}(A)} \mathtt{normalize}_{\mathtt{Ty}}(A) = A^{\mathrm{nfty}}$$

$$\prod_{A:\mathtt{Ty}} \prod_{a:\mathtt{Tm}(A)} \prod_{a^{\mathrm{nftm}}:\mathtt{IsNfTm}(a)} \mathtt{normalize}_{\mathtt{Tm}}(a) = a^{\mathrm{nftm}}$$

by induction on normal forms and neutral terms.

Normalization theorem

Theorem

$\mathtt{IsNfTy}(A)$ *and* $\mathtt{IsNfTm}(a)$ *are contractible.*

This is proved in $\mathbf{Y}$ but stated in $\mathbf{X}$. Since $\mathbf{X} \subset \mathbf{Y}$ is full, this holds also in $\mathbf{X}$.

Coherence problem

Question

Are $\mathrm{Ty}_{\mathbf{I}_\infty(\mathbb{T})}$ *and* $\mathrm{Tm}_{\mathbf{I}_\infty(\mathbb{T})}$ *0-truncated in the* ∞*-topos* $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty}(\mathbb{T}))$*?*

Third working ∞-topos

We go back to $\mathbf{X} = \mathbf{Gl}(\varepsilon^*)$, the Artin gluing for $\varepsilon^* : \mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathbb{T})}) \to \mathbf{Psh}(\mathbf{Ctx}_{\mathbf{R}_\infty(\mathbb{T})})$. From the previous result, we can assume:

Axiom

$\mathtt{IsNfTy}(A)$ and $\mathtt{IsNfTm}(a)$ are contractible.

(At this point we can forget about the normalization model.)

Coherence theorem, internally

TODO

Show

$$\prod_{A:\mathtt{Ty}} \mathtt{IsNfTy}(A) \to \mathtt{IsContr}(\mathfrak{C}(A = A))$$
$$\prod_{A:\mathtt{Ty}} \prod_{a:\mathtt{Tm}(A)} \mathtt{IsNfTm}(a) \to \mathtt{IsContr}(\mathfrak{C}(a = a)).$$

by induction on normal forms and neutral terms.

Theorem

$\mathfrak{C}(\mathtt{Ty})$ *and* $\mathfrak{C}(\mathtt{Tm}(A))$ *are* 0*-truncated.*

Coherence theorem

Theorem

The object $\varepsilon^* \mathrm{Ty}_{\mathbf{I}_\infty(\mathcal{T})}$ *and the map* $\varepsilon^* \mathrm{Tm}_{\mathbf{I}_\infty(\mathcal{T})} \to \varepsilon^* \mathrm{Ty}_{\mathbf{I}_\infty(\mathcal{T})}$ *are* 0*-truncated in* $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{R}_\infty(\mathcal{T})})$.

Lemma

$\varepsilon : \mathbf{Ctx}_{\mathbf{R}_\infty(\mathcal{T})} \to \mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})}$ *is essentially surjective.*

Theorem

The object $\mathrm{Ty}_{\mathbf{I}_\infty(\mathcal{T})}$ *and the map* $\mathrm{Tm}_{\mathbf{I}_\infty(\mathcal{T})} \to \mathrm{Ty}_{\mathbf{I}_\infty(\mathcal{T})}$ *are* 0*-truncated in* $\mathbf{Psh}(\mathbf{Ctx}_{\mathbf{I}_\infty(\mathcal{T})})$.

Conclusion

Summary

Coherence via normalization, using ∞-analogue of relative induction principles and synthetic Tait computability.

- It will work for most type constructors (I checked for Π and some inductive types). How general?
- Part of the proof can/should be formalized in proof assistants. No need to extend/modify type theory: postulating univalence, HITs, and STC axioms is enough.

Related topics

- Coherence theorem (Bidlingmaier 2020; Bocquet 2020, 2021; Curien 1993; Hofmann 1995; Lumsdaine and Warren 2015; Nguyen and Uemura 2022)
- Normalization by evaluation (Altenkirch, Hofmann, and Streicher 1995; Altenkirch and Kaposi 2017; Coquand 2019)
- Synthetic Tait computability[1] (Gratzer 2021; Sterling 2021; Sterling and Angiuli 2021; Sterling and Harper 2021)
- Relative induction principles (Bocquet, Kaposi, and Sattler 2021)
- ∞-topoi and their localizations (Anel et al. 2022; Lurie 2009)
- Internal languages for ∞-topoi (Kapulkin and Lumsdaine 2021; Shulman 2019)
- Modalities in homotopy type theory (Rijke, Shulman, and Spitters 2020)
- Formalization in Coq-HoTT, UniMath, Cubical Agda

[1] https://www.jonmsterling.com/stc-bibliography.html

References I

T. Altenkirch, M. Hofmann, and T. Streicher (1995). “Categorical reconstruction of a reduction free normalization proof”. In: *Category Theory and Computer Science*. Ed. by D. Pitt, D. E. Rydeheard, and P. Johnstone. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 182–199. DOI: 10.1007/3-540-60164-3_27.

T. Altenkirch and A. Kaposi (Jan. 2016). “Type Theory in Type Theory Using Quotient Inductive Types”. In: *SIGPLAN Not.* 51.1, pp. 18–29. DOI: 10.1145/2914770.2837638.

— (Oct. 2017). “Normalisation by Evaluation for Type Theory, in Type Theory”. In: *Logical Methods in Computer Science* 13 (4). DOI: 10.23638/LMCS-13(4:1)2017.

M. Anel, G. Biedermann, E. Finster, and A. Joyal (2022). “Left-exact localizations of ∞-topoi I: Higher sheaves”. In: *Adv. Math.* 400, Paper No. 108268, 64. DOI: 10.1016/j.aim.2022.108268.

References II

M. Artin, A. Grothendieck, and J. L. Verdier (1972-1973). *Théorie des topos et cohomologie étale des schémas*. Vol. 269,270,305. Lecture Notes in Mathematics. Springer.

S. Awodey (2018). "Natural models of homotopy type theory". In: *Mathematical Structures in Computer Science* 28.2, pp. 241–286. DOI: 10.1017/S0960129516000268.

M. E. Bidlingmaier (2020). *An interpretation of dependent type theory in a model category of locally cartesian closed categories*. arXiv: 2007.02900v1.

R. Bocquet (2020). *Coherence of strict equalities in dependent type theories*. arXiv: 2010.14166v1.

— (2021). *Strictification of weakly stable type-theoretic structures using generic contexts*. arXiv: 2111.10862v1.

R. Bocquet, A. Kaposi, and C. Sattler (2021). *Relative induction principles for type theories*. arXiv: 2102.11649v2.

References III

T. Coquand (2019). "Canonicity and normalization for dependent type theory". In: *Theor. Comput. Sci.* 777, pp. 184–191. DOI: 10.1016/j.tcs.2019.01.015.

P.-L. Curien (1993). "Substitution up to Isomorphism". In: *Fundam. Inform.* 19.1/2, pp. 51–85.

P. Dybjer (1996). "Internal Type Theory". In: *Types for Proofs and Programs: International Workshop, TYPES '95 Torino, Italy, June 5–8, 1995 Selected Papers*. Ed. by S. Berardi and M. Coppo. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 120–134. DOI: 10.1007/3-540-61780-9_66.

M. Fiore (2012). *Discrete Generalised Polynomial Functors*. Talk at ICALP 2012. URL: http://www.cl.cam.ac.uk/~mpf23/talks/ICALP2012.pdf.

D. Gratzer (2021). *Normalization for multimodal type theory*. arXiv: 2106.01414v1.

References IV

D. Gratzer, G. A. Kavvos, A. Nuyts, and L. Birkedal (2020). "Multimodal Dependent Type Theory". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. Saarbrücken, Germany: Association for Computing Machinery, pp. 492–506. DOI: 10.1145/3373718.3394736.

R. Harper, F. Honsell, and G. Plotkin (Jan. 1993). "A Framework for Defining Logics". In: *J. ACM* 40.1, pp. 143–184. DOI: 10.1145/138027.138060.

M. Hofmann (1995). "On the interpretation of type theory in locally cartesian closed categories". In: *Computer Science Logic*. Ed. by L. Pacholski and J. Tiuryn. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 427–441. DOI: 10.1007/BFb0022273.

P. T. Johnstone (2002). *Sketches of an Elephant : A Topos Theory Compendium Volume 1*. Vol. 43. Oxford Logic Guides. Oxford University Press.

References V

K. Kapulkin and P. L. Lumsdaine (2021). "The simplicial model of Univalent Foundations (after Voevodsky)". In: *Journal of the European Mathematical Society*. DOI: 10.4171/JEMS/1050.

P. L. Lumsdaine and M. A. Warren (July 2015). "The Local Universes Model: An Overlooked Coherence Construction for Dependent Type Theories". In: *ACM Trans. Comput. Logic* 16.3, 23:1–23:31. DOI: 10.1145/2754931.

J. Lurie (2009). *Higher Topos Theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press. URL: https://www.math.ias.edu/~lurie/papers/HTT.pdf.

H. K. Nguyen and T. Uemura (2022). *∞-type theories*. arXiv: 2205.00798v1.

B. Nordström, K. Petersson, and J. M. Smith (1990). *Programming in Martin-Löf's Type Theory: An Introduction*. Oxford University Press. URL: http://www.cse.chalmers.se/research/group/logic/book/.

References VI

E. Rijke, M. Shulman, and B. Spitters (2020). "Modalities in homotopy type theory". In: *Log. Methods Comput. Sci.* 16.1, Paper No. 2, 79. DOI: 10.23638/LMCS-16(1:2)2020.

M. Shulman (2015). "Univalence for inverse diagrams and homotopy canonicity". In: *Mathematical Structures in Computer Science* 25.05, pp. 1203–1277. DOI: 10.1017/s0960129514000565.

— (2019). *All $(\infty, 1)$-toposes have strict univalent universes*. arXiv: 1904.07004v2.

J. Sterling (2021). "First Steps in Synthetic Tait Computability. The Objective Metatheory of Cubical Type Theory". PhD Thesis. Carnegie Mellon University. URL: https://www.jonmsterling.com/pdfs/sterling:2021:thesis.pdf.

J. Sterling and C. Angiuli (2021). "Normalization for Cubical Type Theory". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1–15. DOI: 10.1109/LICS52264.2021.9470719.

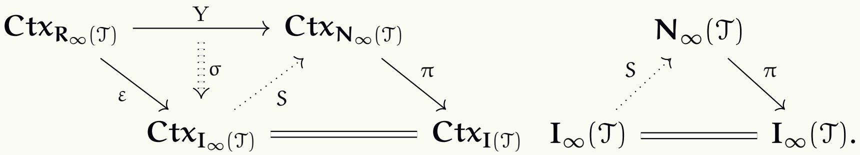
References VII

J. Sterling and R. Harper (2021). "Logical Relations as Types: Proof-Relevant Parametricity for Program Modules". In: *J. ACM* 68.6, 41:1–41:47. DOI: 10.1145/3474834.

V. Voevodsky (2015). "A C-system defined by a universe category". In: *Theory and Applications of Categories* 30.37, pp. 1181–1214.

Normalization model

We have an externalization $\mathbf{N}_\infty(\mathcal{T})$ of the internal normalization model, and it is equipped with a projection $\pi : \mathbf{N}_\infty(\mathcal{T}) \to \mathbf{I}_\infty(\mathcal{T})$ and a section $Y : \mathbf{Ctx}_{\mathbf{R}_\infty(\mathcal{T})} \to \mathbf{Ctx}_{\mathbf{N}_\infty(\mathcal{T})}$ over ε. The relative induction principle gives



Oplax limits over inverse categories

Oplax limits over inverse categories (Shulman 2015) are generalized/iterated gluing.

Example (cf. Elephant A4.5.5)

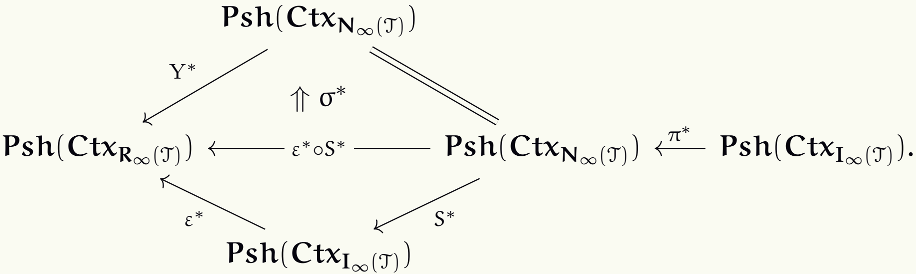
Let I be a finite poset and $\mathcal{X}$ an ∞-topos. $\mathcal{X}^{\mathrm{I}}$ is the oplax limit of $\mathrm{I}^{\mathrm{op}} \ni \_ \mapsto \mathcal{X} \in \mathbf{Cat}$.

1. $\mathcal{X}^{\mathrm{I}}$ is an ∞-topos.
2. For any upward-closed subset $\mathrm{J} \subset \mathrm{I}$, we have a subterminal $\mathrm{P}_{\mathrm{J}} \in \mathcal{X}^{\mathrm{I}}$ defined by $\mathrm{P}_{\mathrm{J}}(i) = 1$ if $i \in \mathrm{J}$ and $\mathrm{P}_{\mathrm{J}}(i) = 0$ otherwise.
3. $\mathcal{X}^{\mathrm{J}}$ is the open subtopos associated to P_{J}
4. $\mathcal{X}^{\mathrm{I} \setminus \mathrm{J}}$ is the closed subtopos associated to P_{J}.

So any object of $\mathcal{X}^{\mathrm{I}}$ can be fractured into subdiagrams.

Second working ∞-topos

Our second working ∞-topos **Y** is the oplax limit of



Y contains a lot of modalities, and everything we need can be axiomatized.

Multimodal type theory?

Bocquet, Kaposi, and Sattler (2021) use *multimodal type theory* (Gratzer et al. 2020) to explain normalization proof.

- Interpretation of MTT in diagrams of ∞-topoi is not clear. We would have to strictify functors and natural transformations as well as ∞-topoi.
- I don't know if MTT has been implemented. STC is ready to formalize in existing proof assistants.

Normalization vs higher normalization

Normalization for $\mathbf{I}_\infty(\mathcal{T})$ does not directly imply normalization for $\mathbf{I}(\mathcal{T})$.

- After proving $\mathbf{I}_\infty(\mathcal{T}) \simeq \mathbf{I}(\mathcal{T})$, we have normalization for $\mathbf{I}(\mathcal{T})$.
- The normalization model $\mathbf{N}_\infty(\mathcal{T})$ constructed using higher STC is not set-valued, so we don't have $\mathbf{I}(\mathcal{T}) \to \mathbf{N}_\infty(\mathcal{T})$ before the coherence theorem.