

# Semantics for two-dimensional type theory

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## 2-Dimensional interpretations of type theory

There are many interpretations of type theory that are 2-dimensional in a certain sense

- ▶ The groupoid interpretation by Hofmann and Streicher<sup>1</sup>
- ▶ The two-dimensional models by Garner<sup>2</sup>

Interpreted in something like **groupoids**

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<sup>1</sup>Hofmann, Martin, and Thomas Streicher. "The groupoid interpretation of type theory." *Twenty-five years of constructive type theory (Venice, 1995)* 36 (1998): 83-111.

<sup>2</sup>Garner, Richard. "Two-dimensional models of type theory." *Mathematical structures in computer science* 19.4 (2009): 687-736.

# Directed type theory

But directed variants have also been considered

- ▶ An interpretation with directed definitional equality<sup>3</sup>
- ▶ A syntactical framework for directed type theory<sup>4</sup>
- ▶ An interpretation with directed identity types<sup>5</sup>

Interpreted in something like **categories**

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<sup>3</sup>Licata, Daniel R., and Robert Harper. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.

<sup>4</sup>Nuyts, Andreas. Towards a directed homotopy type theory based on 4 kinds of variance. Master's thesis, KU Leuven, 2015.

<sup>5</sup>North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

# A framework is missing

## **Problem:**

- ▶ Garner gave a general notion of 2-dimensional comprehension category, but this only works for **undirected** type theory
- ▶ The interpretations of directed type theory are ad hoc

## **Goal of this talk:**

*find categorical framework in which one can interpret various flavors of 2-dimensional type theory*

The work in this talk has been formalized using UniMath.

# Idea

- ▶ Use bicategories instead of categories
- ▶ Define **comprehension bicategories**.
- ▶ For that, we need a bicategorical notion of fibration
- ▶ We use ingredients by Hermida<sup>6</sup> and by Buckley<sup>7</sup>
- ▶ Find suitable instances of comprehension bicategories

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<sup>6</sup>Hermida, Claudio. "Some properties of Fib as a fibred 2-category." *Journal of Pure and Applied Algebra* 134.1 (1999): 83-109.

<sup>7</sup>Buckley, Mitchell. "Fibred 2-categories and bicategories." *Journal of Pure and Applied Algebra* 218.6 (2014): 1034-1074.

# Comprehension categories

Type theory can be interpreted in **comprehension categories**.

## Definition

A **comprehension category** is a *strictly* commuting triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\chi} & \mathcal{C} \rightarrow \\ & \searrow F & \swarrow \text{cod} \\ & \mathcal{C} & \end{array}$$

where  $F$  is a Grothendieck fibration and where  $\chi$  preserves cartesian cells.

# Fibrations of bicategories

- ▶ The notion of fibration of bicategories has a **global** condition and **local** condition.

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- ▶ The notion of fibration of bicategories has a **global** condition and **local** condition.
- ▶ For the global and local condition, we need an appropriate notion of cartesian cell.
- ▶ We also require that cartesian 2-cells are closed under horizontal composition.

## The global condition

A pseudofunctor satisfying the global condition is called a **global fibration**.

The global condition basically says the following: given

- ▶ contexts  $\Gamma_1$  and  $\Gamma_2$
- ▶ a substitution  $s : \Gamma_1 \rightarrow \Gamma_2$
- ▶ a type  $A$  in context  $\Gamma_2$

we get a type  $A[s]$  in context  $\Gamma_1$ .

## The local condition

We think of 2-cells  $\tau : s_1 \Rightarrow s_2$  as reductions from  $s_1$  to  $s_2$

A pseudofunctor satisfying the local condition is called a **local opfibration**.

## The local condition

We think of 2-cells  $\tau : s_1 \Rightarrow s_2$  as reductions from  $s_1$  to  $s_2$   
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**Local condition:** given

- ▶ contexts  $\Gamma_1, \Gamma_2$
- ▶ substitutions  $s_1, s_2 : \Gamma_1 \rightarrow \Gamma_2$
- ▶ a reduction  $\tau : s_1 \Rightarrow s_2$
- ▶ a type  $A$  in context  $\Gamma_2$
- ▶ a term  $t$  of type  $A[s_1]$

we get a a term of type  $A[s_2]$ .

# Comprehension bicategories

A **comprehension bicategory** is a *strictly* commuting triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\chi} & \mathcal{B} \rightarrow \\ & \searrow F & \swarrow \text{cod} \\ & \mathcal{B} & \end{array}$$

where  $\chi$  preserves cartesian cells and where  $F$  is a global fibration and a local opfibration.

## Examples of comprehension bicategories

Given a **locally groupoidal** bicategory  $\mathcal{B}$  with pullbacks, take

$$\begin{array}{ccc} \mathcal{B}^{\rightarrow} & \xrightarrow{\text{id}} & \mathcal{B}^{\rightarrow} \\ & \searrow \text{cod} & \swarrow \text{cod} \\ & \mathcal{B} & \end{array}$$

## Examples of comprehension bicategories

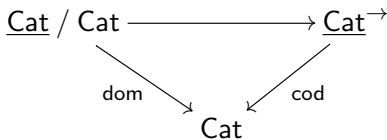
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This does **not** work for arbitrary bicategories.

## Examples of comprehension bicategories

The work by North<sup>8</sup> and by Licata and Harper<sup>9</sup> is encapsulated in the following comprehension bicategory



Here:

- ▶ Cat is the bicategory of categories
- ▶ Cat is the category of categories

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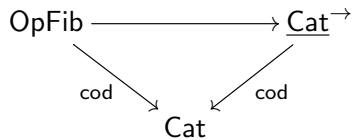
<sup>8</sup>North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

<sup>9</sup>Licata, Daniel R., and Robert Harper. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.



## Examples of comprehension bicategories

We have the following comprehension bicategory



Categorification of the previous one

## Examples of comprehension bicategories

Given a bicategory  $\mathcal{B}$  with pullbacks, take

$$\begin{array}{ccc} \text{SOpFib}(\mathcal{B}) & \xrightarrow{\quad} & \mathcal{B}^{\rightarrow} \\ & \searrow \text{cod} & \swarrow \text{cod} \\ & & \mathcal{B} \end{array}$$

Here  $\text{SOpFib}(\mathcal{B})$  is the bicategory of Street opfibrations in  $\mathcal{B}$ .

## Remark on the formalization

- ▶ The notion of comprehension bicategory and fibration have been formalized using UniMath.
- ▶ Here we make use of **displayed bicategories**<sup>10</sup>

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<sup>10</sup>Ahrens, B., et al (2022). Bicategories in univalent foundations. *Mathematical Structures in Computer Science*, 1-38.

# Conclusion

- ▶ We defined a notion of **comprehension bicategory**
- ▶ This is a suitable framework in which one can interpret (directed) type theory: we proved **soundness**
- ▶ There are general instances of this definition (internal Street fibrations)

Further work: look at type formers, completeness